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Chern-Simons field theory on a spatial torus surface

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Introduction

Chern-Simons theory is a pure gauge theory in three dimensions; this model is interesting not only *per se*, but also for its applications in Mathematics and in Physics. In the mathematical context, the Chen-Simons theory can be used to compute topological invariants. In Physics, phenomenological models —based on the Chern-Simons Lagrangian terms— have been produced to describe the dynamics of quasi-particles with odd statistics in particular solid state systems.

From a theoretical point of view, Chern-Simons theory is interesting for its peculiar characteristics as a topological gauge field theory. In fact, studying this theory, we have the chance to face closely and to manage the main features of the path-integral formalism.

Indeed, the Chern-Simons action can be written as a three-form integrated over the whole three-dimensional manifold, for this reasons the theory can be classically defined even if no metric is given, this feature is called *general covariance*. Even if, at the quantum level, one has to introduce a gauge fixing that explicitly breaks the general covariance, it can be proved that the observables maintain this invariance.

An important expectation value which can be deduced from the theory is the *Wilson loop*, which corresponds to the trace of the holonomy computed along a closed path in the space. This expectation value is, indeed, a gauge invariant and general covariant quantity. Thus, the Wilson loops connect the Chern-Simons theory and the knot theory, because they represent topological invariants associated with knots and links.

Chern-Simons theory has originally be defined in $\mathbb{R}^3(S^3)$ and then it can be extended to all closed, connected and orientable three-dimensional manifolds. So far, this nontrivial extension to a generic 3-manifold M has been made by producing a surgery transformation —and its corresponding operator realization— acting on the observables of $\mathbb{R}^3$. This mathematical construction is not based on a direct path-integral computation in M.

In this thesis we attempt to define directly the path-integral and the perturbative method in a particular manifold, i.e. the three-dimensional torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ with gauge structure group $SU(2)$. The above mentioned manifold represents a simple example of a nontrivial manifold with Abelian fundamental group $\pi_1(\mathbb{T}^3) = \mathbb{Z}^3$. We present a general introduction to the Chern-Simons theory on $\mathbb{T}^3$ which is based on the perturbative approach defined with respect to a nontrivial stationary point of the action. Stationary points of the Chern-Simons action correspond to flat connections on $\mathbb{T}^3$, whose holonomies determine an $SU(2)$ representation of the fundamental group. A complete classification of the flat connections in this manifold is possible because one can easily determine all the $SU(2)$ representations of $\pi_1(\mathbb{T}^3)$. The perturbative method in $\mathbb{T}^3$ presents problems which are related with the presence of zero modes, *i.e.* modes which do not appear in the quadratic part of the expansion of the action in powers of the fields. When the background field is vanishing (modulo gauge), zero modes are elements of the de Rham cohomology which has dimension equal to $3 \times$ (dimension of the algebra) $= 9$. But when the background field is non-vanishing, zero modes are elements of the cohomology associated with the covariant derivative; it is shown that, in this case, the zero-modes space has dimension equal to $3 \times 1 = 3$. Therefore the perturbative expansion performed around a nontrivial background flat connection minimizes the number of non-propagating degrees of freedom. The perturbative computations presented in this thesis have been performed by integrating on all the degrees of freedom with the exception of zero modes.

In the first chapter, we make a general introduction to the field theory in the path-integral approach. We define the generating functionals of the fields and the perturbative expansion. At the end we elaborate the gauge theory and on the BRST method

In the second chapter, we concentrate on the Chern-Simons theory as a field theory in $\mathbb{R}^3$. We introduce both the Abelian and the non-Abelian Chern-Simons action; in particular, we describe the non-Abelian Chern-Simons theory which is the main object of this thesis work. We show how to fix the gauge and we define the Wilson loop, which produces a complete set of observables of the theory. Moreover, we discuss the relation between the Chern-Simons observables and the invariant of the Knot theory in $\mathbb{R}^3$.

The original part of this thesis starts from chapter 3. The goal is to define the path-integral in the Chern-Simons theory in the manifold $\mathbb{T}^3 = S^1 \times S^1 \times S^1$. We look into the flat connections which are the solutions of the equations of motion and we specify an explicit representation of all the flat connection. We describe a procedure to reduce the non-propagating degrees of freedom and to define a perturbative approach in spite of the persistent zero modes, based on the background method. Moreover, we introduce a suitable basis for the fields in the background approach, which displays an analogy with a sort of $U(1)$ charge conservation. We compute the propagator of all the field variables orthogonal to the zero modes and the expression of the vertices.

In the fourth chapter, we define and compute the *partition function* to the second order for the fields which do not contain the zero mode. We produce the evidence of the cancellation of a wide class of diagrams at all orders by using the manifested analogy with the charged fields. Furthermore, the produced decomposition in the field variables allows to maintain under control the zero modes.

In the fifth chapter, a general method for the computation of the Wilson loop in the theory is developed. We compute the expectation values of the Wilson line operator, which are associated with typical knots, to the lower order of the perturbation theory. For a certain class of knots, we demonstrate that the linking number coincides with the result in the $\mathbb{R}^3$ case.

Chapter 1

Path integral quantization

This chapter is a brief review of the *path-integral method* in Quantum Field Theory [1]-[2]. Our goal here is to give an introduction to this subject and fix the conventions. Thus, in this first part we introduce the notations and we emphasize some aspects of the basic theory which we find essential in further discussions.

The path-integral method in quantum mechanics provides an equivalent formulation to the Schrödinger's and the Heisenberg's ones. In the framework of the field theory the path-integral method is particularly convenient to compute the expectation values of the observables and to explore the main features of a model. In addition, when the fields are defined in a manifold which does not admit a decomposition $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ refers to time, the canonical approach is not known.

The essential brick of the path-integral is the action functional written as the integral of the Lagrangian density in the whole manifold

$$S[\phi^a(x)] = \int \mathrm{d}^n x\, \mathcal{L}(\phi^a(x), \partial_\mu \phi^a(x)) \tag{1.1}$$

where $\phi^a(x)$ represents a generic field with some discrete indices a.

Classically, one is interested on the *on-shell* fields, *i.e.* fields that verify the Eulero-Lagrange equations of motion

$$\frac{\delta S}{\delta \phi^a} = \frac{\delta \mathcal{L}}{\delta \phi^a} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^a} = 0\,.$$

These equations represent a stationary condition for the action with respect to the variations $\delta\phi^a$ of the fields (with fixed boundary conditions). The set of the free (*off-shell*) values of the fields will be denoted by the *configuration space*.

In the path-integral method the quantization is performed by imposing that the expectation value of each observable $\mathcal{O}$ (functional of the fields) is equal to the following quantity:

$$\langle \mathcal{O} \rangle \equiv \frac{\int [\mathrm{D}\phi]\, e^{iS[\phi]} \mathcal{O}[\phi]}{\int [\mathrm{D}\phi]\, e^{iS[\phi]}}, \tag{1.2}$$

where the integral has to be carried out over all the off-shell configurations of the fields. Expression (1.2) is not really a ratio of two well defined quantities, but it has to be understood as the limit, in which the number N of degrees of freedom goes to infinity, of a ratio of standard integrals with fixed N. Namely, if one assumes that the fields can be written as a linear combination of a complete orthonormal set $\{\phi_n(x)\}$

$$\phi(x) = \sum_n^{\infty} c_n \phi_n(x),$$

one can cut off the sum to a finite number N of terms and define the differential

$$\mathrm{d}\phi(x) = \sum_n^{N} \mathrm{d}c_n \phi_n(x).$$

In this framework, the functional differential at fixed N can be written as finite product:

$$[\mathrm{D}\phi]_N = \prod_n^N \mathrm{d}c_n;$$

and one can set a suitable interpretation of the expression (1.2) by means of the following limit:

$$\langle \mathcal{O} \rangle \equiv \lim_{N\to\infty} \frac{\int [\mathrm{D}\phi]_N \; e^{iS[\phi]}\mathcal{O}[\phi]}{\int [\mathrm{D}\phi]_N \; e^{iS[\phi]}}, \tag{1.3}$$

where for fixed N both the numerator and the denominator are reduced to standard integral in the coefficient c_n. An important tool allowing to derive other interesting quantities from this definition is the *functional derivative* defined by the following basic axiom

$$\frac{\delta}{\delta J^a(x)} J^b(y) = \delta^{ab}\delta^n(x-y), \tag{1.4}$$

where n is the dimension of the manifold and $J^a(x)$ a generic set of functions.

Example: Quadratic operator

Let us look into the fundamental case in which the functional integral has the form:

$$I(J^a) = \int_{-\infty}^{\infty} \prod_n^N \mathrm{d}c_n e^{-\frac{1}{2}c_a Q^{ab} c_b + J^a c_a}, \tag{1.5}$$

where Q^{ab}, J^a and k are real numbers for all $a, b = 1, \cdots, N$, and c_n are real variables. This kind of integral appear in expression (1.3) when the Lagrangian is at most quadratic in the fields. Actually, since in expression (1.3) there is the imaginary exponential of the action, the integral (1.5) is obtained by means of an analytic continuation. Since the operator Q, whose coefficients are represented by Q^{ab}, has to be symmetric and positive defined, it can be diagonalized by the adjoint action of orthogonal matrix S

$$D = SQS^{-1} = SQS^{t} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix},$$

where λ_a are the eigenvalues of Q with respect to the eigenvectors $v_a = S^{ab} c_b$. Since by definition S is orthogonal, the relative Jacobian is trivial and the product of differentials with respect to the new variables is $\prod_n^N \mathrm{d}v_n = \prod_n^N \mathrm{d}c_n$. Therefore, if we define $J' = SJ$, the integral (1.5) can be written as

$$\begin{aligned} I(J^a) &= \int_{-\infty}^{\infty} \prod_n^N \mathrm{d}v_n e^{-\frac{1}{2} v_a D^{ab} v_b + J^a (S^{ab})^t v_b} = \prod_n^N \int_{-\infty}^{\infty} \mathrm{d}v_n e^{-\frac{1}{2}\lambda_n (v_n)^2 + J'^n v_n} = \\ &= \prod_n^N \int_{-\infty}^{\infty} \mathrm{d}v_n e^{-\frac{\lambda_n}{2}(v_n)^2 + J'^n v_n} = \prod_n^N e^{\frac{(J'^n)^2}{2\lambda_n}} \int_{-\infty}^{\infty} \mathrm{d}v_n \, e^{-\left(\sqrt{\frac{\lambda_n}{2}} v_n - \frac{J'}{\sqrt{2\lambda_n}}\right)^2} \\ &= \prod_n^N \sqrt{\frac{2}{\lambda_n}} e^{\frac{J'^n (D^{-1})_{nn} J'^n}{2}} \int_{-\infty}^{\infty} \mathrm{d}v'_n \, e^{-(v'_n)^2} = e^{\frac{1}{2} J'^t D J'} \prod_n^N \sqrt{\frac{2\pi}{\lambda_n}} = \\ &= \sqrt{(2\pi)^N} e^{\frac{1}{2} J^t S^{-1} D^{-1} S J} \sqrt{\frac{1}{\prod_n^N \lambda_n}} = \sqrt{\frac{(2\pi)^N}{\mathrm{Det} Q}} e^{\frac{1}{2} J^t Q^{-1} J}. \end{aligned} \tag{1.6}$$

□

1.1 Functionals

The expectation value expressed in (1.2) cannot be exactly evaluated in most cases: essentially it can be done only if the integrand is quadratic, as in the previous example. In most cases, the observables are general functions of the fields and the action is not merely quadratic, but it has the so-called interaction terms. Let us consider first *free theories*, *i.e.* no interaction terms in the action,

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2} \phi(x) \mathbf{Q} \phi(x).$$

where $\mathbf{Q}$ stands for a differential operator and ϕ for a generic bosonic[1] field (or a set of fields).

Assuming that each observable can be expanded in powers of the field, in calculating quantities like (1.2), one can concentrate the investigation in finding the

[1] With some care, this formalism can be extended to the fermionic fields case in a rather standard way.

expectation values of the correlation function

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle \;=\; \frac{\int[\mathrm{D}\phi]\; e^{iS[\phi]}\phi(x_1)\cdots\phi(x_n)}{\int[\mathrm{D}\phi]\; e^{iS[\phi]}} \,.$$

Looking at the integrand, one can easily realize that all these quantities can be derived from the so-called *generating functional*

$$Z[J] \equiv \frac{\int[\mathrm{D}\phi]\; e^{iS[\phi]+i\int \mathrm{d}^n x\, J(x)\phi(x)}}{\int[\mathrm{D}\phi]\; e^{iS[\phi]}} \tag{1.7}$$

where a so-called *source term* has been added to the Lagrangian, where J is an auxiliary field called *source*. Using the definition of the functional derivative one can write

$$\langle \phi(x_1)\cdots\phi(x_n)\rangle \;=\; \left(-i\frac{\delta}{\delta J(x_1)}\right)\cdots\left(-i\frac{\delta}{\delta J(x_n)}\right) Z[J]\Bigg|_{J=0} \,. \tag{1.8}$$

Now defining the propagator[2], $G(x-y)$, as the Green function of the quadratic operator

$$\mathbf{Q}\, G(x-y) = i\delta(x-y)\,,$$

one can complete the square in the exponent of the generating functional by introducing a shifted field

$$\phi'(x) = \phi(x) - i\int \mathrm{d}^n x\, G(x-y)J(y) \Rightarrow \phi(x) = \phi'(x) + i\int \mathrm{d}^n y\, G(x-y)J(y)\,.$$

Thus the above-mentioned quantity becomes

$$\begin{aligned}\int \mathrm{d}^n x\,\frac{1}{2}\phi(x)\mathbf{Q}\phi(x) \quad &+\quad \int \mathrm{d}^n x\, J(x)\phi(x) = \\ = \int \mathrm{d}^n x\,\frac{1}{2}\phi'(x)\mathbf{Q}\phi'(x) \quad &-\quad \int \mathrm{d}^n x\,\mathrm{d}^n y\,\frac{1}{2}J(x)G(x-y)J(y)\,.\end{aligned} \tag{1.9}$$

This substitution is a translation and it is assumed that the *functional measure* transforms trivially, *i.e.* $[D\phi]=[D\phi']$. In the last expression ϕ' and J are decoupled. Thus, one can rewrite equation (1.7) bringing the propagator exponential out of the integral, one finds

$$Z[J] = e^{-i\int \mathrm{d}^n x\,\mathrm{d}^n y\,\frac{1}{2}J(x)G(x-y)J(y)}\,.$$

Note that expression (1.8) of the correlation function indeed fulfils the important property known as the *Wick's theorem*, which states

$$\begin{aligned}\langle \phi(x_1)\cdots\phi(x_{2n})\rangle \;&=\; \sum_i G(x_1-x_i) < \phi(x_2)\cdots\hat{\phi}(x_i)\cdots\phi(x_{2n}) > = \\ &=\; \sum_p G\left(x_{p(1)}-x_{p(2)}\right)\cdots G\left(x_{p(2n-1)}-x_{p(2n)}\right) \\ \langle \phi(x_1)\cdots\phi(x_{2n+1})\rangle \;&=\; 0\,,\end{aligned} \tag{1.10}$$

[2]One assumes that the propagator depends only on the difference $(x-y)$ by virtue of the translation invariance.

namely the even points correlation function is the sum of a product of propagators over all inequivalent ways to match pairwise the points[3], while the odd ones vanish.

Now it is easy to extend these expressions to the theories with an interaction term by simply expanding its exponential

$$e^{iS_I[\phi]} = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} S_I^n[\phi].$$

It consists of the formal perturbative expansion. Writing the generating functional as in expression (1.7)

$$\begin{aligned}
Z[J] &= Z^{-1} \int [\mathrm{D}\phi]\, e^{iS_0[\phi]+iS_I[\phi]+i\int \mathrm{d}^n x\, J(x)\phi(x)} = \\
&= Z^{-1} \int [\mathrm{D}\phi] \left(\sum_{n=0}^{\infty} \frac{(i)^n}{n!} S_I^n[\phi] \right) e^{iS[\phi]+i\int \mathrm{d}^n x\, J(x)\phi(x)} = \\
&= \left(\sum_{n=0}^{\infty} \frac{(i)^n}{n!} S_I^n \left[\frac{\delta}{\delta J} \right] \right) \int [\mathrm{D}\phi]\, e^{iS_0[\phi]+i\int \mathrm{d}^n x\, J(x)\phi(x)} = \\
&= \left(\sum_{n=0}^{\infty} \frac{(i)^n}{n!} S_I^n \left[\frac{\delta}{\delta J} \right] \right) Z_0[J]\,,
\end{aligned} \tag{1.11}$$

we introduce the zero subscript to indicate the free parts. Each term of the expansion can be graphically represented by the well-known formalism of the *Feynman diagrams*. Let us recall the definitions of *connected diagram* and *one-particle-irreducible diagram* which we use in what follows. The first one obviously stands for a diagram where all vertices are connected. One-particle-irreducible denotes diagrams which remain connected if one "cuts" an internal line. Actually, one can note that the generating functional is even redundant, in fact it can be written as the exponential of another functional $W[J]$, that account only for the connected diagrams

$$Z[J] = e^{i\,W[J]}.$$

Moreover, with a Legendre transformation of the $W[J]$, a new functional can be introduced, which is finally the main object to study the perturbative expansion and it is called the *effective action* $\Gamma[\Phi]$. Here Φ is a new classical field defined as

$$\frac{\delta W[J]}{\delta[J(x)]} = \Phi(x)$$

that could be solved finding

$$J(x) = \hat{J}[\Phi](x).$$

[3]The hat in the first row means that one has to remove that factor.

The above mentioned Legendre transformation is then

$$\Gamma[\Phi] = W[\hat{J}] - \int \mathrm{d}^n x \, \Phi(x)\hat{J}(x).$$

This functional takes into account only the one particle irreducible diagrams, but one can obtain all the correlation functions from it.

Effective action can be expanded in powers of Φ

$$\Gamma[\Phi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int \mathrm{d}^n \, x_1 \cdots \mathrm{d}^n \, x_n \Gamma^(n)(x_1, \cdots, x_n)\Phi(x_1) \cdots \Phi(x_n),$$

where each term has n external lines and $\Gamma^{(n)}(x_1, \cdots, x_n)$ is called a *proper vertex*. Furthermore it can be expanded in number of loops that coincides with an expansion in powers of $\hbar$.

The main feature of the effective action is indeed that it contains all the relevant physical information. It means that the physical amplitudes of the model can be derived from it. However, in order to do that, one has to specify the observed values of the parameters of the theory. All the observables of the theory depends on certain number of parameters: these parameters must be defined in term of observed quantities. This procedure is performed by defining the *normalization conditions*:

- **Mass**
  the mass, m, of a particle corresponds to the pole of the dressed propagator, namely the zero of the quadratic part of the effective action

$$\Gamma^{(2)}(k^2)\Big|_{k^2=m^2} = 0;$$

- **Wave function normalization**
  the zero above mentioned has to be simple

$$\frac{\partial \Gamma^{(2)}(k^2)}{\partial k^2}\Bigg|_{k^2=m^2} = 1;$$

- **Coupling constant**
  the value of the coupling constant, g, can be identified with the limit of the appropriate proper vertex as the external momenta approach zero

$$\lim_{k_i \to 0} \Gamma^{(n)}(k_1, \cdots, k_n) = -g,$$

  where $n = 3$ or $n = 4$ depending on the form of the interaction lagrangian.

The key idea is that the "bare" parameters, namely those that appear in the Lagrangian, generally do not coincide with the physical quantities that are determined by the proper vertices.

Normalization conditions are not trivially satisfied in the loop expansion of perturbation theory and the previous idea become fundamental, since some diagram could diverge. To save the day one must impose the validity of the normalization condition at each order of perturbation theory.

Before that, one has to characterize the divergences by a regularization procedure, that is performed for example introducing a *cut-off*[4] in the extremes of the integral, which will be sent to infinity at the end. Once the nature of a divergence of a diagram is known, it is possible to add to the original Lagrangian a counterterm that cancels the divergence. In fact, by differentiating with respect to the masses and to the external momenta of the divergent integrals, it can be demonstrated that the counterterms are local, which is a necessary property for them to be added to the Lagrangian. Thus, counterterms are essentially polynomials of the fields and their derivatives defined at the same point. Note that, since the counterterms are infinite quantities, one can add an arbitrary finite term with the same power of field and its derivatives. Different choices of these finite quantities are called *subtraction schemes* and do not change physical observables.

More generally one can add local counterterms in order to eliminate the divergences, in order to maintain the symmetries of the theory (eventually broken by the regularization) and in order to maintain the normalization conditions.

If the forms of all the counterterms that preserve the symmetries of the theory are already in the Lagrangian, the theory is called renormalizable. At each loop order, one can find a redefinition of the fields and of the coupling constants in such a way that the counterterms are included in the bare quantities. The bare parameters, as said, are not observable, therefore this procedure is consistent and well-defined.

Actually, the regularized divergences depend on a energy scale parameter, μ, thus the renormalized quantities are also functions of it. The *beta function* and the *anomalous dimension* are functions used to take into account this scale dependence respectively of the coupling constants and of the fields.

Viceversa if the form of some symmetry-preserving counterterms is not yet in the Lagrangian, one can consider the theory in which a term of such form is added *a priori*, namely before the quantization. Sometimes this procedure is not stable and for each term we add *a priori* in the original Lagrangian, a new divergence arises which needs in turn a counterterm of a new form.

Composite fields

Composite fields can be understood as combination of monomials of fields and their derivatives defined at the same point. One has to care about them because they do not renormalize in a straightforward way, namely not as the same product with

[4]There are several regularization methods that can be more appropriated. We will not discuss them here, note that each method generally breaks some property of the theory, like the gauge symmetries or unitarity, the latter in the *cut-off* regularization.

factors defined in different points. To deal with a product of fields at coincident points one has to define properly the meaning of the composite field. Generally they are managed by introducing suitable extra source terms in the generating functional. In the following chapter we show that in the Chern-Simons theory composite fields enter the definition of the *Wilson line* and that there is a convenient definition of this kind of composite fields which guarantees the general covariance.

1.2 Gauge theories

Gauge theories are the cornerstone of the standard model, the actual theory which describe particle interactions thus far. Each particle is represented by a field that transforms in a precise way under a local symmetry associated to a gauge group and the action is invariant under gauge transformations. The interactions are “mediated” by particular bosonic fields, the connections, that take value in the algebra of the gauge group. The gauge invariance, required for the whole action, restricts the observables to those quantities that do not change under gauge transformations.

It is well-known that a few problems arise when trying to quantize the connections part of the action. Canonically, one can observe that this action derives from constrained system, namely the Lagrangian is *singular*. That means

$$\det\left(\frac{\delta^2\mathcal{L}}{\delta\dot{\phi}_i\dot{\phi}_j}\right)=0,$$

which implies that the momenta

$$\Pi_i=\frac{\delta\mathcal{L}}{\delta\dot{\phi}_i}$$

are not all independent. This fact prevents from fixing the canonical quantization law

$$[\Pi_i,\phi_j]=i\hbar\delta_{ij}.$$

In general the problem is that, due to the invariance of the action with respect of the local gauge transformations, there are redundant degrees of freedom that are not dynamical, *i.e.* physical. In fact the connection field configuration space is divided in equivalence classes, the so-called gauge orbits, and there is no “propagation” between elements of the same class,[13]. As a consequence of this, one finds that the quadratic part of the action is not invertible, so the propagator cannot be find and the perturbative expansion cannot be performed. In the path-integral formalism the “measure” is ill-defined because it leads to a divergent multiplicative factor coming from the volume of the gauge orbits.

Among the ways to solve this problem, in this thesis, we shall use the BRST method.

1.2.1 BRST method

For the sake of clarity we prefer to introduce the BRST method, [4]-[5], directly by its application to a gauge theory with $SU(N)$ as structure group. We consider an action functional of the vector potential A_μ which is invariant under gauge transformation.

A_μ is an element of the algebra $su(N)$. It is well known that the latter has N^2-1 generators T^a with $a = 1,..,n = N^2 - 1$ which satisfy the following commutation rules

$$[T^a, T^b] = i f_{abc} T^c$$

where f_{abc} are called *structure constants.* Each element of the group and the vector potential can be written as

$$g(x) = e^{i\epsilon(x)} = e^{i\epsilon^a(x)T^a} \ , \quad A_\mu = A_\mu^a T^a \ .$$

Defining the *covariant derivative* as $\mathrm{D}_\mu \cdot \equiv \partial_\mu \cdot + i[A_\mu, \cdot]$ and imposing that it has to be covariant under the gauge transformations, one obtains the transformation of the vector potential under a generic gauge element

$$A'_\mu = g^{-1} A_\mu g - i g^{-1} \partial_\mu g, \tag{1.12}$$

which corresponds to the variation

$$\delta_\epsilon A_\mu(x) = \mathrm{D}_\mu \epsilon(x) \ , \tag{1.13}$$

if one takes the gauge parameter ϵ infinitesimal. Note that this transformation has the structure of a commutator *closed algebra*, which means that if one takes two infinitesimal parameters ϵ and χ, there exists a third parameter $\xi = [\epsilon, \chi]$, such that

$$[\delta_\epsilon, \delta_\chi] A_\mu = \delta_\xi A_\mu. \tag{1.14}$$

One can introduce an anticommuting parameter θ and a fermionic field $C(x)$, called *ghost field*, in order to rewrite the gauge transformations as global transformations:

$$\delta_{\mathrm{BRST}} A_\mu(x) = \theta \, \mathrm{D}_\mu C(x) \ ; \tag{1.15}$$

$$\delta_{\mathrm{BRST}} C(x) = \frac{1}{2} i \, \theta \, [C(x), C(x)] \ ; . \tag{1.16}$$

Due to the closed algebra property, the above defined variation operator is nilpotent.

In fact, by applying δ'_{BRST} to the equation (1.15), one obtains:

$$\begin{aligned}
\delta'_{\text{BRST}}\ (\theta\, \mathrm{D}_\mu C) &= \theta\, \delta'_{\text{BRST}}\ (\partial_\mu C + i[A_\mu, C]) = \\
&= \frac{1}{2} i\, \theta\, \theta'\, \partial_\mu [C, C] + i\theta\, \theta'\, [\mathrm{D}_\mu C, C] - \frac{1}{2}\theta\, \theta' [A_\mu, [C, C]] \ = \\
&= \frac{1}{2} i\, \theta\, \theta'\, \partial_\mu [C, C] + i\theta\, \theta'\, [\partial_\mu C, C] - \\
&\quad -\theta\, \theta'\, [[A_\mu, C], C] - \frac{1}{2}\theta\, \theta' [A_\mu, [C, C]] \ = \\
&= \frac{1}{2} i\, \theta\, \theta'\, \partial_\mu (C^a C^b)[T^a, T^b] + i\theta\, \theta'\, (\partial_\mu C^a) C^b [T^a, T^b] - \\
&\quad -\theta\, \theta'\, A^a_\mu C^b C^c \left([[T^a, T^b], T^c] - \frac{1}{2}[[T^b, T^c], T^a] \right) = 0\,,
\end{aligned}$$

where in the last step one uses the anticommuting property of the ghost field $C^a C^b = -C^b C^a$ and the *Jacobi identity*:

$$[T^a, T^b], T^c] + [T^b, T^c], T^a] + [T^c, T^a], T^b] = 0.$$

Whereas by applying δ'_{BRST} to the (1.16), one obtains:

$$\begin{aligned}
\delta'_{\text{BRST}} \left(\frac{1}{2} i\, \theta\, [C, C] \right) &= -\frac{1}{4}\theta\ \left([\theta'[C, C], C] + [C, \theta'[C, C]] \right) = \\
&= -\frac{1}{4}\theta\ \left([\theta'[C, C], C] + [\theta'[C, C], C] \right) = \\
&= -\frac{1}{2}\theta\ \left([\theta'[C, C], C] \right) = -\frac{1}{2}\theta\theta'\ [[C, C], C] = \\
&= -\frac{1}{2}\theta\theta' C^a C^b C^c\ [[T^a, T^b], T^c] = 0\,,
\end{aligned}$$

where in the last step we use $C^a C^b C^c = C^b C^c C^a = C^c C^a C^b$ and the Jacobi identity. Moreover, two other fields are introduced in this formalism. The auxiliary field B, with bosonic statistic, and the antighost $\bar{C}$ which instead is fermionic. They form a so-called *conjugated pair*, since the BRST variations on their sector are trivially nilpotent. Thus, the complete set of BRST transformations is:

$$\delta_{\text{BRST}} A_\mu(x) = \theta\, \mathrm{D}_\mu C(x)\,; \tag{1.17}$$

$$\delta_{\text{BRST}} C(x) = \frac{1}{2} i\, \theta\, [C(x), C(x)]\,; \tag{1.18}$$

$$\delta_{\text{BRST}}\, \bar{C} = \theta\, B\,; \tag{1.19}$$

$$\delta_{\text{BRST}}\, B = 0\,. \tag{1.20}$$

It has to be underling that, since the BRST variation on the original fields reduce itself to a gauge transformation, gauge invariant functionals of the original fields are by construction BRST invariant too.

In this framework, the main idea of the BRST method is to modify the action in such a way that the resulting quadratic operator in the Lagrangian becomes invertible. To make it possible, it is necessary to break the gauge symmetry, but one can carry out this operation leaving the BRST symmetry intact. Moreover, the modification of the action has to be performed such that the expectation values of the gauge invariant observables remain unvaried. These reasonable requests are fulfilled by adding to the action a proper BRST *exact term*, namely a functional of the fields which is a BRST variation of an other functional. In fact, assuming that one has modified the original action by an exact term, the so-called *gauge fixing*, which guarantees the existence of the propagator, the resulting action is trivially BRST invariant. Thus, one has only to verify that the expectation values of the observables calculated with the new action coincide with the results in the original theory. Let λ be a parameter which vary continuously in the interval $[0, 1]$ and let us modify the action as follows

$$S \to S + \lambda S_{GF},$$

where the gauge fixing is exact, *i.e.* $S_{GF} = \delta_{\mathrm{BRST}} X$, such that at $\lambda = 0$ we obtain the original action, whereas at $\lambda = 1$ we obtain the modified action that we are looking for. An observable is defined as a generic gauge invariant function of the original fields and we want to show that its expectation values of a generic observable, do not depend on the value of λ, namely

$$\partial_\lambda \left\langle \mathcal{O} \right\rangle\big|_\lambda = 0 \;\; \forall \lambda \in [0, 1].$$

Since the modified action is BRST invariant for each value of λ, the expectation value of a total BRST variation vanishes, for each value of λ. Moreover, being the observables gauge invariant, their BRST variation vanish. Therefore, we obtain

$$\begin{aligned}
\partial_\lambda \left\langle \mathcal{O} \right\rangle\big|_\lambda &= \left\langle i S_{GF} \mathcal{O} \right\rangle - \left\langle i S_{GF} \right\rangle \left\langle \mathcal{O} \right\rangle = \\
&= \left\langle i \delta_{\mathrm{BRST}} X \mathcal{O} \right\rangle - \left\langle i \delta_{\mathrm{BRST}} X \right\rangle \left\langle \mathcal{O} \right\rangle = \\
&= i \left\langle \delta_{\mathrm{BRST}} (X \mathcal{O}) \right\rangle - i \left\langle X \delta_{\mathrm{BRST}} \mathcal{O} \right\rangle - i \left\langle \delta_{\mathrm{BRST}} X \right\rangle \left\langle \mathcal{O} \right\rangle = 0 \,.
\end{aligned}$$

Actually, often one use the name BRST transformation in regard to the equation (1.17)-(1.20) simplifying the parameter θ. Thus one obtain the operator s

$$\begin{aligned}
s A_\mu(x) &= \mathrm{D}_\mu C(x) \,; && (1.21) \\
s C(x) &= \frac{1}{2} i \left[C(x), C(x) \right] ; && (1.22) \\
s \bar{C} &= B \,; && (1.23) \\
s B &= 0 \,, && (1.24)
\end{aligned}$$

which is nilpotent and anticommuting, namely send commuting fields in anticommuting field and *viceversa*. Thus, the gauge fixing can be deduced from the so-called gauge fermion by applying s. It is important to say that, since its anticommuting feature, s fulfils the following Leibniz rule

$$s(AB) = s(A)B + (-)^{\chi} A s(B),$$

where $\chi = \{0, 1\}$ represent the statistic of the field A, respectively if it is bosonic or fermionic. In general the gauge fermion have the following form

$$\Psi = \int \mathrm{d}^n x \, \mathrm{tr} \left(\bar{C} \left[G(A) + \frac{\lambda}{2} B \right] \right), \tag{1.25}$$

where λ is a dimensional parameter and $G(A)$ is an opportune function of the vector potential. Whence the total action which allows to carry out the calculation of the expectation values is:

$$S_{\text{TOT}} = S + s(\Psi).$$

Chapter 2

Chern-Simons as a field theory

In general, the Chern-Simons theory is a gauge theory in three dimensions in which the action only depends on A that is a connection with respect to the structure group G; see [20]-[26] and references therein. Without loss of generality, we can assume $G = U(1)$ or $G = SU(N)$. The Chern-Simons theory in $\mathbb{R}^3$, as in pure QED or in pure Yang-Mills theory in $\mathbb{R}^4$, is characterized by an action which only depends on the vector potential A_μ[1]. Just as in pure QED or Yang-Mills theory, the gauge invariance prevents us from performing a direct quantization of the theory. As seen in the previous chapter, the BRST method solves this problem.

For the sake of clarity we collect all the steps of standard calculus in the appendix at the end of the chapter.

2.1 Abelian Chern-Simons theory

Let us consider the case in which U(1) is the structure group. A general element of U(1) in a point x can be written as

$$g(x) = e^{i\epsilon(x)}.$$

The generic gauge transformation of the vector potential is

$$A'_\mu = A_\mu - ig^{-1}\partial_\mu g = A_\mu + \partial_\mu \epsilon. \tag{2.1}$$

The action of the Abelian Chern-Simons theory is

$$S_{cs} = \frac{k}{8\pi} \int_{\mathbb{R}^3} \mathrm{d}^3x \, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \, , \tag{2.2}$$

where $\varepsilon^{\mu\nu\rho}$ is the completely antisymmetric tensor called *Levi-Civita tensor*. Note that in four dimensions one cannot build a term of the Lagrangian contracting

[1]We mean that, since the principal bundle of the group $U(1)$, respectively $SU(N)$, is trivial in $\mathbb{R}^3$, the connection A can be represented by a one-form $A_\mu \mathrm{d}x^\mu$ well defined in the whole manifold.

this invariant tensor with only the partial derivative and the vector potential. The Abelian field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which represents the curvature of the connection.

The solutions of the equations of motion are the *flat connections*, in fact

$$\frac{\delta S_{cs}}{\delta A_\mu} = \frac{k}{8\pi}\varepsilon^{\mu\nu\rho}\,(\partial_\nu A_\rho - \partial_\rho A_\nu) = \frac{k}{8\pi}\varepsilon^{\mu\nu\rho}\,F_{\nu\rho} = 0 \Leftrightarrow F_{\nu\rho} = 0.$$

It is easy to realize that the action (2.2) is invariant under transformations (2.1) considering that partial derivatives commute, while Levi-Civita tensor is antisymmetric:

$$\begin{aligned} \delta S_{cs} &= \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}\,[\partial_\mu\epsilon\,\partial_\nu A_\rho + A_\mu\partial_\nu\partial_\rho\epsilon] = \\ &= -\frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}\,\epsilon\,\partial_\mu\partial_\nu A_\rho = 0 \quad \forall\,\epsilon\,. \end{aligned}$$

In the Abelian Chern-Simons theory the BRST transformations deduced from the variations (2.1) are

$$s\,A_\mu = \partial_\mu C, \quad s\,\bar{C} = B,$$

$$s\,C = 0, \qquad s\,B = 0$$

and we can choose as gauge fermion

$$\begin{aligned} \Psi &= -\frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\,\bar{C}\partial_\mu A_\mu \;\Rightarrow \\ \Rightarrow s\Psi &= \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\,\left[-B\partial_\mu A_\mu + \bar{C}^a\partial_\mu\partial_\mu C\right] = \\ &= \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\,\left[\frac{1}{2}(A_\mu\partial_\mu B - B\partial_\mu A_\mu) - \partial_\mu\bar{C}\partial_\mu C\right]. \end{aligned}$$

This choice corresponds to the Landau gauge. The use of a covariant gauge of general type (for instance, the Feynman gauge) necessitates the introduction of a dimensional parameter, which would break the scale invariance. At the end we obtain the total action

$$S_{tot} = \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\,\left[\frac{1}{2}\,\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2}\,A_\mu\partial_\mu B - \frac{1}{2}\,B\partial_\mu A_\mu - \partial_\mu\bar{C}\partial_\mu C\right]. \tag{2.3}$$

With few steps of algebra one finds the propagator of the vector potential

$$\langle A_\mu(x)A_\nu(y)\rangle = -\frac{i}{k}\varepsilon_{\mu\rho\nu}\partial^\rho\frac{1}{|x-y|} = \frac{i}{k}\varepsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}\,; \tag{2.4}$$

$$\langle A_\mu(x)B(y)\rangle = -\frac{i}{k}\frac{(x-y)_\mu}{|x-y|^3}\,; \tag{2.5}$$

$$\langle B(x)A_\mu(y)\rangle = \frac{i}{k}\frac{(x-y)_\mu}{|x-y|^3}\,; \tag{2.6}$$

$$\left\langle C(x)\bar{C}(y)\right\rangle = -\frac{i}{k}\frac{1}{|x-y|}\,. \tag{2.7}$$

2.2 Linking number

The expression (2.4) is rather peculiar,[19]. In fact, up to a multiplicative factor, it appear in the definition of the *Gauss linking integral*:

$$\varphi(K_1, K_2) = \frac{1}{4\pi} \oint_{K_1} \mathrm{d}x^\mu \oint_{K_2} \mathrm{d}y^\nu \varepsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}, \tag{2.8}$$

where the variables x^μ and y^ν run on the knot K_1 and the knot K_2 respectively(in the appendix we carry out the integral calculation). The above quantity does not depend on the parametrization of the knots and does not change under smooth deformation of the knots. To sum up, it is a topological invariant. It is an integer which "counts" how many times K_2 winds around K_1 or *viceversa.*

2.3 Non-Abelian Chern-Simons theory

As in standard Yang-Mills theory, let $SU(N)$ be the gauge group of the system. Using the definition of the generators given in the previous chapter, the vector potential and the field strength tensor, that are now matrices in the algebra $su(N)$, can be written as linear combinations of the generators:

$$A_\mu = A_\mu^a T^a, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f_{abc} A_\mu^b A_\nu^c) T^a = F_{\mu\nu}^a T^a.$$

The non-Abelian Chern-Simons action is

$$\begin{aligned} S_{cs} &= \frac{k}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\, \mathrm{tr}\left[A_\mu \partial_\nu A_\rho + i\frac{2}{3} A_\mu A_\nu A_\rho \right] = \\ &= \frac{k}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho} \frac{1}{2} \left(A_\mu^a \partial_\nu A_\rho^a - \frac{1}{3} f_{abc} A_\mu^a A_\nu^b A_\rho^c \right), \end{aligned} \tag{2.9}$$

where in the last step we fix $\mathrm{tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$ as normalization. The equations of motion still have as solutions the null-curvature connections

$$\frac{\delta S_{cs}}{\delta A_\mu^a} = \frac{k}{8\pi} \varepsilon^{\mu\nu\rho} \left(\partial_\nu A_\rho^a - \partial_\rho A_\nu^a - f_{abc} A_\nu^b A_\rho^c \right) = \frac{k}{8\pi} \varepsilon^{\mu\nu\rho}\, F_{\nu\rho}^a = 0 \;\Leftrightarrow\; F_{\nu\rho}^a = 0\,.$$

For convenience, let us rewrite here the transformation (1.12) of the vector potential under a generic gauge element, $g(x) = e^{i\epsilon(x)}$,

$$A'_\mu = g^{-1} A_\mu g - i g^{-1} \partial_\mu g,$$

which corresponds infinitesimally to the variation (1.13)

$$\delta A_\mu = \mathrm{D}_\mu \epsilon\,.$$

Actually, in the non-Abelian case the original action (2.9) is not gauge invariant and its variation is

$$\delta S_{cs} = -i \frac{k}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho} \partial_\mu \mathrm{tr}[A_\nu (\partial_\rho g) g^{-1}] + 2\pi k S_{WZ}[g]. \tag{2.10}$$

Because of the total derivative, the first term is a boundary one and it vanishes when we assume null infinity conditions on the field A_μ. The second term is called *Wess-Zumino term* and it is explicity

$$S_{WZ}[g] = \frac{1}{24\pi^2}\int_{\mathbb{R}^3} \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\, \mathrm{tr}\left[g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\rho g\right]. \tag{2.11}$$

This term takes values in $\mathbb{Z}$ and depends only on the homotopy class of the gauge transformation map $g(x)$. In the path-integral formulation one is interested in quantities where the action appears as argument of the imaginary exponential, *i.e.* e^{iS}, thus fixing $k \in \mathbb{Z}$ the contribution of S_{WZ} to the expectation values of the theory vanishes.

Beside the above global reason, only the infinitesimal variation are concerned in the perturbative calculation and the BRST method concerns only the gauge transformation connected to the identity. Therefore, in order to guarantee the consistency of the perturbative approach, it is enough to verify that $S_{WZ}[g]$ is invariant under infinitesimal continuous deformation of the gauge transformation map $g(x)$.

The BRST transformations deduced from the infinitesimal variations (1.13) are:

$$s\, A_\mu = \mathrm{D}_\mu C, \qquad s\, \bar{C} = B,$$

$$s\, C = -\tfrac{i}{2}[C, C], \quad s\, B = 0\,.$$

Once again, a convenient choice for the gauge fermion is the Landau one

$$\begin{aligned}
\Psi &= -\frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\, \bar{C}^a \partial_\mu A^a_\mu \;\Rightarrow \qquad (2.12)\\
\Rightarrow s\Psi &= \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\, \left[-B^a\partial_\mu A^a_\mu + \bar{C}^a\partial_\mu(\mathrm{D}_\mu C)^a\right] = \\
&= \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\, \left[\frac{1}{2}(A^a_\mu\partial_\mu B^a - B^a\partial_\mu A^a_\mu) - \partial_\mu\bar{C}^a(\mathrm{D}_\mu C)^a\right].
\end{aligned}$$

Therefore, the total action is

$$\begin{aligned}
S_{tot} &= \frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\, \left[\frac{1}{2}\,\varepsilon^{\mu\nu\rho}A^a_\mu\partial_\nu A^a_\rho + \frac{1}{2}\,A^a_\mu\partial_\mu B^a - \frac{1}{2}\,B^a\partial_\mu A^a_\mu\right] - \\
&\quad -\frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\, \left[\partial_\mu\bar{C}^a\partial_\mu C^a\right] + \\
&\quad +\frac{k}{4\pi}\int_{\mathbb{R}^3} \mathrm{d}^3x\, \left[f_{abc}\partial_\mu\bar{C}^a A^b_\mu C^c - \frac{1}{6}\varepsilon^{\mu\nu\rho}f_{abc}A^a_\mu A^b_\nu A^c_\rho\right].
\end{aligned} \tag{2.13}$$

Again, with few steps of algebra, one finds the propagators

$$\left\langle A^a_\mu(x)A^b_\nu(y)\right\rangle = -\frac{i}{k}\delta^{ab}\varepsilon_{\mu\rho\nu}\partial^\rho\frac{1}{|x-y|} = \frac{i}{k}\delta^{ab}\varepsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}\,; \tag{2.14}$$

$$\left\langle A^a_\mu(x)B^b(y)\right\rangle = -\frac{i}{k}\delta^{ab}\frac{(x-y)_\mu}{|x-y|^3}\ ; \tag{2.15}$$

$$\left\langle B^a(x)A^b_\mu(y)\right\rangle = \frac{i}{k}\delta^{ab}\frac{(x-y)_\mu}{|x-y|^3}\ ; \tag{2.16}$$

$$\left\langle C^a(x)\bar{C}^b(y)\right\rangle = -\frac{i}{k}\delta^{ab}\frac{1}{|x-y|}\ . \tag{2.17}$$

As shown in the previous chapter, because of the presence of the interaction term, one has to enquire about the consistency of the perturbative expansion of the theory, which immediately turns out to be renormalizable by *power counting*.

Moreover, it has been proved [6]-[8], that the theory is not only renormalizable but also finite, thus both the beta *function* and the *anomalous dimensions* vanish.

2.4 General covariance

The peculiar characteristic of the Chern-Simons theory is that the original action does not depend on the metric $g_{\mu\nu}$ that one could introduce on the manifold. In fact it is easy to see that each term of the Lagrangian has all the indices of the fields and of the partial derivatives contracted with the antisymmetric Levi-Civita invariant tensor. This is a consequence of the fact that the action can be written as a three-form integrated in the whole three-dimensional manifold

$$S = \frac{k}{4\pi}\int_M \mathrm{tr}\left[A\wedge \mathrm{d}A + i\,\frac{2}{3}A\wedge A\wedge A\right]\ ,$$

where A is the one-form $A = A_\mu \mathrm{d}x^\mu$. An immediate consequence of the independence on the metric is that the *energy-momentum tensor* $T_{\mu\nu}$, vanishes because

$$T^{\mu\nu} \propto \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} = 0.$$

It means that the states of this theory have zero energy and vanishing momentum. It has been argued [14]-[16] that the number of these states depends on the topology features of the manifold and in $\mathbb{R}^3$ there is only one ground state. Note that in the BRST procedure we introduced a metric in the gauge fermion (2.12), $\partial_\mu A^a_\mu = g^{\mu\nu}\partial_\mu A^a_\nu$,

$$i.e.\quad g_{\mu\nu} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}_{\mu\nu}\ .$$

However, this terms does not perturb the physical space, indeed the *energy-momentum tensor* $\Theta_{\mu\nu}$, of the total action (2.13) can be written as a BRST variation

$$\Theta_{\mu\nu} = \frac{k}{4\pi}\left[\mathcal{Q}, \frac{k}{4\pi}(\partial_{[\mu}\bar{C}^a A^a_{\nu]} - g_{\mu\nu}\partial_\rho\bar{C}^a A^a_\rho)\right]\ ,$$

where $\mathcal{Q}$ is the BRST charge operator. Since $\mathcal{Q}$ annihilates the physical states, it means that the expectation value of $\Theta_{\mu\nu}$ still vanishes on this space.

2.5 Wilson loop

From now on we specialize in the non-Abelian three-dimensional Chern-Simons theory which is the main subject of this thesis.

The gauge invariance and the general covariance impose stringent constraints to the observables. An important gauge invariant quantity of any gauge theory, is the *Wilson loop*, which in this topological theory acquires more importance because of its relation with the knots and links. To define this quantity let us introduce the holonomy operator along a given path γ, parametrized as

$$\{x(t) : t \in [0,1] \text{ with } \gamma(0) = x_1,\ \gamma(1) = x_2\}\,,$$

with a fixed irreducible representation ρ of the Lie algebra of G:

$$U_\rho(x_1, x_2; \gamma) = \mathrm{P}\, e^{i\int_\gamma \mathrm{d}x^\mu A_\mu^{(\rho)}(x)} = \mathrm{P}\, e^{i\int_0^1 \mathrm{d}t\, \dot{x}^\mu(t) A_\mu^{(\rho)}(x(t))}\,, \tag{2.18}$$

where P means the path-ordered prescription, so the previous expression can be written as an exponential expansion as follows

$$\begin{aligned} U_\rho(x_1, x_2; \gamma) &= \mathrm{P}\sum_{n=0}^{\infty} \frac{i^n}{n!}\left(\int_0^1 \mathrm{d}t\, \dot{x}^\mu(t) A_\mu^{(\rho)}(x(t))\right)^n = \\ &= \mathbb{I} + i\int_0^1 \mathrm{d}t\, \dot{x}^\mu(t) A_\mu^{(\rho)}(x(t)) + \\ &\quad + i^2 \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \dot{x}^\mu(t)\, \dot{x}^\nu(t')\, A_\nu^{(\rho)}(x(t')) A_\mu^{(\rho)}(x(t)) + \cdots \end{aligned} \tag{2.19}$$

Note that the holonomy is an element of an unitary irreducible representation ρ of the gauge group and since the exponential is a one-form integrated in a one-dimensional space, it is well-defined and does not depend on the metric. Under a gauge transformation g of the connection (1.12), the holonomy changes covariantly

$$U'_\rho(x_1, x_2; \gamma) = g^{-1}(x_1) U_\rho(x_1, x_2; \gamma) g(x_2),$$

thus in order to obtain a gauge invariant quantity one can consider the trace of the holonomy along a closed path $C \in \{\gamma(t) | \gamma(0) = \gamma(1)\}$, namely the Wilson loop

$$W_\rho(C) = \mathrm{tr}[U_\rho(x, x; C)] = \mathrm{tr}\left[\mathrm{P}\, e^{i\oint_C \mathrm{d}x^\mu A_\mu^{(\rho)}(x)}\right]. \tag{2.20}$$

Using the cyclic property of the trace, under a gauge transformation g, one obtains

$$W'_\rho(C) = \mathrm{tr}[U'_\rho(x, x; C)] = \mathrm{tr}[g^{-1}(x) U_\rho(x, x; C) g(x)] = \mathrm{tr}[U_\rho(x, x; C)] = W_\rho(C).$$

Therefore, the Wilson loop is general covariant, as well as $U_\rho(x_1, x_2; \gamma)$, and a gauge invariant quantity.

In the perturbative framework one considers the expansion of the Wilson loop

$$W_\rho(C) = \mathrm{tr}\left[\mathrm{P}\sum_{n=0}^{\infty}\frac{i^n}{n!}\left(\oint_C \mathrm{d}x^\mu A_\mu^{(\rho)}(x)\right)^n\right] = \sum_{n=0}^{\infty} i^n\, W_\rho^{(n)}(C). \tag{2.21}$$

More generally, expression (2.20) can be straightforwardly extended to the links, which are disjoint unions of knots.

Due to the general covariance and the gauge invariance, as it has been first suggested in [14] by Witten, expression 2.21 evaluated on a link $\mathcal{L}$ contains terms whose proper collections have expectation values related to some topological invariants of $\mathcal{L}$.

2.6 Framing and self-linking number

One can easily realize that the terms $W_\rho^{(n)}(C)$ with n higher than one contain composite operators. Consequently, $W_\rho(C)$ also is a composite operator. As it has been stated in the previous chapter, each composite operator needs a specific definition in order to give it a well-defined expectation value. As we will see, the expectation values containing expression (2.19) have ambiguities, that must be removed.

There is a particular definition of the $W_\rho^{(n)}(C)$ in the Chern-Simons theory which is really suitable and meaningful for maintaining general covariance. To understand the features of this definition is sufficient to consider the quadratic term $W_\rho^{(2)}(C)$

$$W_\rho^{(2)}(C) = \mathrm{tr}\left[\mathrm{P}\left(\frac{1}{2}\oint_C \mathrm{d}x^\mu A_\mu^{(\rho)}(x)\right)^2\right]\,, \tag{2.22}$$

which is a composite operator because the integrand contains the product of two vector potential evaluated in the same point. Indeed, expression (2.22) reads

$$W_\rho^{(2)}(C) \;\equiv\; \mathrm{tr}\left[\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \dot{x}^\mu(t)\, \dot{x}^\nu(t')\, A_\nu^{(\rho)}(x(t'))A_\mu^{(\rho)}(x(t))\right]\,. \tag{2.23}$$

To understand why expression (2.23) leads to ambiguities, let us evaluate its expectation value:

$$\begin{aligned}
\left\langle W_\rho^{(2)}(C)\right\rangle &= \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \dot{x}^\mu(t)\, \dot{x}^\nu(t')\, \mathrm{tr}\left[< A_\nu^{(\rho)}(x(t'))A_\mu^{(\rho)}(x(t)) >\right] = \\
&= \frac{i}{k}\dim(\rho)C_2(\rho)\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \dot{x}^\mu(t)\, \dot{x}^\nu(t')\, \varepsilon_{\mu\nu\rho}\frac{(x(t)-x(t'))^\rho}{|x(t)-x(t')|^3} = \\
&= \frac{2\pi i}{k}\dim(\rho)C_2(\rho)\varphi(C)\,,
\end{aligned}$$

where $C_2(\rho)$ the quadratic Casimir invariant of the representation ρ with the normalization fixed by

$$T^aT^a = C_2(\rho)\mathbb{I} \quad \text{and} \quad \mathrm{tr}[T^aT^b] = \frac{\dim(\rho)}{\dim(G)}C_2(\rho)\delta^{ab}. \tag{2.24}$$

Whereas $\varphi(C)$ stands for the function

$$\begin{aligned}\varphi(C) &= \frac{1}{2\pi}\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \dot{x}^\mu(t)\, \dot{x}^\nu(t')\, \varepsilon_{\mu\nu\rho} \frac{(x(t)-x(t'))^\rho}{|x(t)-x(t')|^3} = \\ &= \frac{1}{4\pi}\int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}t' \dot{x}^\mu(t)\, \dot{x}^\nu(t')\, \varepsilon_{\mu\nu\rho} \frac{(x(t)-x(t'))^\rho}{|x(t)-x(t')|^3}.\end{aligned}$$

Even if the integrand is divergent when $t = t'$, the whole integral is finite. Unfortunately, as it has been shown by Calugareanu [18], $\varphi(C)$ is metric dependent and is not invariant under deformations of C, *i.e.* $\varphi(C)$ is not topological invariant.

In order to maintain the general covariance, the "correct" way to define the $W^{(n)}_\rho(C)$ is by introducing the *framing*. Roughly speaking, one defines a new closed path C_f in a tubular neighbourhood of C and performs the integrals separately along the two closed paths. To be more precise let us introduce an infinitesimal parameter χ. Starting from the parametrization $C = \{x^\mu(t) | t \in [0,1] \wedge x^\mu(0) = x^\mu(1)\}$, one can produce the parametrization of a new closed path as

$$C_f = \{y^\mu(t) = x^\mu(t) + n^\mu(t)\chi | t \in [0,1]; y^\mu(0) = y^\mu(1)\}\,. \tag{2.25}$$

where n^μ is a normal vector which can be taken orthogonal to $\dot{x}^\mu$ for each value of t and can wind around C while t goes along it. Then, one can insert expression (2.25) into the definition (2.23),

$$W^{(2)}_\rho(C)\Big|_f \equiv \lim_{\chi\to 0}\left(\mathrm{tr}\left[\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \dot{x}^\mu(t)\, \dot{y}^\nu(t')\, A^{(\rho)}_\nu(y(t'))A^{(\rho)}_\mu(x(t))\right]\right)\,.$$

The new expectation value of this quantity is

$$\begin{aligned}\left\langle W^{(2)}_\rho(C)\Big|_f\right\rangle &= \lim_{\chi\to 0}\frac{i}{k}\dim(\rho)C_2(\rho)\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \dot{x}^\mu(t)\, \dot{y}^\nu(t')\, \varepsilon_{\mu\nu\rho} \frac{(x(t)-y(t'))^\rho}{|x(t)-y(t')|^3} = \\ &= i\,\frac{2\pi}{k}\dim(\rho)C_2(\rho)\varphi_f(C)\,,\end{aligned} \tag{2.26}$$

where $\varphi_f(C)$ is defined as

$$\begin{aligned}\varphi_f(C) &= \frac{1}{4\pi}\lim_{\chi\to 0}\int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}t'\, \dot{x}^\mu(t)\, \dot{y}^\nu(t')\, \varepsilon_{\mu\nu\rho} \frac{(x(t)-y(t'))^\rho}{|x(t)-y(t')|^3} = \\ &= \frac{1}{4\pi}\oint_C \mathrm{d}x^\mu \oint_{C_f} \mathrm{d}y^\nu\, \varepsilon_{\mu\nu\rho} \frac{(x(t)-y(t'))^\rho}{|x(t)-y(t')|^3}\,,\end{aligned} \tag{2.27}$$

that is called the *self-linking number* of C; it is a topological invariant integer defined for framed knots. In the last step we use the fact that the argument of the limit does not depend on χ. However, $\varphi(C)_f$ depends on the choice of $n^\mu(t)$ and it coincides with the linking number defined in expression (2.8) between the knot C and its fixed framing C_f. To extend this definition to all terms $W_\rho^{(n)}(C)$ one can introduce $(n-1)$ new closed paths with the same procedure of framing:

$$C_f^i = \{y_i^\mu(t) = x_i^\mu(t) + n^\mu(t)\chi_i | t \in [0,1]; y_i^\mu(0) = y_i^\mu(1)\} \tag{2.28}$$

with $i = 1, \cdots, n-1$ and $\chi_1 < \chi_2 < \cdots < \chi_{n-1}$. Note that we impose the same orthogonal vector n^μ. Thus, the framing definition of the $W_\rho^{(n)}(C)$ can be written as follows

$$\begin{aligned} W_\rho^{(n)}(C)\Big|_f \quad &\equiv \quad \lim_{\chi_i \to 0} \Bigg\{ \mathrm{tr} \Bigg[\int_0^1 \mathrm{d}t \, \dot{x}^\mu \int_0^t \mathrm{d}t_1 \, \dot{x_1}^{\mu_1} \cdots \\ &\cdots \int_0^{t_{n-2}} \mathrm{d}t_{n-1} \, \dot{x}_{n-1}^{\mu_{n-1}} A_{\mu_{n-1}}(x(t_{n-1})) \cdots A_{\mu_1}(x(t_1)) A_{\mu_1}(x(t_1)) \Bigg] \Bigg\} \, . \end{aligned}$$

2.7 Appendix

2.7.1 Propagator

Generally, the propagator $\langle \phi^a(x)\phi^b(y) \rangle$ for a generic field is the causal solution of the Green equation for the quadratic differential operator $\mathbf{Q}$ of the action, that we call Γ for the bosonic fields and Λ for the fermionic fields. Thus, one has to deduce the propagator from the equation

$$\mathbf{Q} \left\langle \phi^a(x)\phi^b(y) \right\rangle = i \, \delta^{ab} \delta^3(x-y). \tag{2.29}$$

First, let us consider the bosonic case. To write compactly the equation we are interested in, we introduce the total vector field $\Omega_i^a = (A_\mu, B)$ with $i = 0, 1, 2, 3 = \mu, 3$. Now the bosonic quadratic part of the action can be written as follows:

$$\begin{aligned} S_{\Omega\Omega} \quad &= \quad \frac{k}{4\pi} \int_{\mathbb{R}^3} \mathrm{d}^3x \, \left[\frac{1}{2} \, \varepsilon^{\mu\nu\rho} A_\mu^a \partial_\nu A_\rho^a + \frac{1}{2} \, A_\mu^a \partial_\mu B^a - \frac{1}{2} \, B^a \partial_\rho A_\rho^a \right] = \\ &= \quad \int_{\mathbb{R}^3} \mathrm{d}^3x \, \frac{1}{2} \Omega_i^a \left[\frac{k}{4\pi} \delta_{ab} \begin{pmatrix} \varepsilon^{\mu\nu\rho}\partial_\nu & \partial_\mu \\ -\partial_\rho & 0 \end{pmatrix}_{ij} \right] \Omega_j^b = \\ &= \quad \int_{\mathbb{R}^3} \mathrm{d}^3x \, \frac{1}{2} \Omega_i^a \, \Gamma_{ab}^{ij} \, \Omega_j^b \, . \end{aligned} \tag{2.30}$$

Thus, the equation to be solved is

$$\Gamma_{ac}^{ik} \left\langle \Omega_k^c(x) \Omega_j^b(y) \right\rangle = i \delta^{ab} \delta_{ij} \delta^3(x-y) \Rightarrow$$

$$\Rightarrow \begin{pmatrix} \varepsilon^{\mu\nu\rho}\partial_\nu & \partial_\mu \\ -\partial_\rho & 0 \end{pmatrix}_{ik} \begin{pmatrix} \langle A_\rho^a(x) A_\sigma^b(y) \rangle & \langle A_\rho^a(x) B^b(y) \rangle \\ \langle B^a(x) A_\sigma^b(y) \rangle & \langle B^a(x) B^b(y) \rangle \end{pmatrix}_{kj} = i \frac{4\pi}{k} \delta^{ab} \delta_{ij} \delta^3(x-y) \, .$$

Indeed, one can separate this equation with respect to the spatial indices and one obtains the following system:

$$\begin{cases} \varepsilon^{\mu\nu\rho}\partial_\nu \left\langle A^a_\rho(x)A^b_\sigma(y)\right\rangle + \partial_\mu \left\langle B^a(x)A^b_\sigma(y)\right\rangle = i\frac{4\pi}{k}\delta^{ab}\delta_{\mu\sigma}\delta^3(x-y) \\ \varepsilon^{\mu\nu\rho}\partial_\nu \left\langle A^a_\rho(x)B^b(y)\right\rangle + \partial_\mu \left\langle B^a(x)B^b(y)\right\rangle = 0 \\ -\partial_\rho \left\langle A^a_\rho(x)A^b_\sigma(y)\right\rangle = 0 \\ -\partial_\rho \left\langle A^a_\rho(x^\mu)B^b(x^\mu)\right\rangle = i\frac{4\pi}{k}\delta^{ab}\delta^3(x-y)\,. \end{cases} \tag{2.31}$$

To find directly the solution of this system one writes the propagators as Fourier anti-transformation of unknown functions of the momentum p. After imposing some symmetry restrictions one obtains their general expressions:

$$\begin{aligned} \left\langle A^a_\rho(x)A^b_\sigma(y)\right\rangle &= \delta^{ab}\int \frac{d^3p}{(2\pi)^3}e^{ip\cdot(x-y)}\left[\delta_{\rho\sigma}D(p^2)+p_\rho p_\sigma E(p^2)+\varepsilon_{\rho\sigma\tau}p_\tau F(p^2)\right]\;; \\ \left\langle A^a_\rho(x)B^b(y)\right\rangle &= \delta^{ab}\int \frac{d^3p}{(2\pi)^3}e^{ip\cdot(x-y)}p_\rho G(p^2)\,; \\ \left\langle B^a(x)A^b_\sigma(y)\right\rangle &= \delta^{ab}\int \frac{d^3p}{(2\pi)^3}e^{ip\cdot(x-y)}(-)p_\sigma G(p^2)\,; \\ \left\langle B^a(x)B^b(y)\right\rangle &= \delta^{ab}\int \frac{d^3p}{(2\pi)^3}e^{ip\cdot(x-y)}H(p^2)\,. \end{aligned}$$

Therefore, in the momentum representation, where partial derivative becomes a multiplication operator, *i.e.* $\partial = ip$, the differential system (2.31) turns into an algebraic one:

$$\begin{cases} \varepsilon^{\mu\nu\rho}ip_\nu\left[\delta_{\rho\sigma}D(p^2)+p_\rho p_\sigma E(p^2)+\varepsilon_{\rho\sigma\tau}p_\tau F(p^2)\right]+ip_\mu(-p_\sigma)G(p^2) = i\frac{4\pi}{k}\delta_{\mu\sigma} \\ \varepsilon^{\mu\nu\rho}ip_\nu p_\rho G(p^2)+ip_\mu H(k^2) = ip_\mu H(k^2) = 0 \Rightarrow H(p^2)=0 \\ -ip_\rho\left[\delta_{\rho\sigma}D(p^2)+p_\rho p_\sigma E(p^2)+\varepsilon_{\rho\sigma\tau}p_\tau F(p^2)\right] = 0 \\ -ip_\rho p_\rho G(p^2) = i\frac{4\pi}{k} \Rightarrow G(p^2) = -\frac{4\pi}{k}\frac{1}{p^2}\,. \end{cases} \tag{2.32}$$

In the second equation the first term vanishes because $\varepsilon^{\mu\nu\rho}$ is antisymmetric while $p_\nu p_\rho$ *viceversa* is indeed symmetric. There are terms of the same type in the first and the third equation. The latter is reduced to

$$\begin{aligned} 0 &= -ip_\rho\delta_{\rho\sigma}D(p^2) - ip_\rho p_\rho p_\sigma E(p^2) - i\varepsilon_{\rho\sigma\tau}p_\rho p_\tau F(p^2) = -ip_\sigma D(p^2) - ip^2 p_\sigma E(p^2) \Rightarrow \\ &\Rightarrow D(p^2) = -p^2E(p^2)\,. \end{aligned}$$

Finally, replacing the previous results in the first equation of (2.32) and separating the symmetric and antisymmetric part with respect to the indices μ and σ, one obtains

$$\begin{aligned}
\frac{4\pi}{k}[\delta_{\mu\sigma} - \frac{p_\mu p_\sigma}{p^2}] &= \varepsilon^{\mu\nu\sigma} p_\nu p^2 E(p^2) + \varepsilon^{\mu\nu\rho} p_\nu p_\rho p_\sigma E(p^2) + \varepsilon^{\mu\nu\rho}\varepsilon_{\rho\sigma\tau} p_\nu p_\tau F(p^2) = \\
&= \varepsilon^{\mu\nu\rho}[p_\nu p^2 \delta_{\rho\sigma} + p_\nu p_\rho p_\sigma] E(p^2) + (\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}) p_\nu p_\tau F(p^2) \Rightarrow
\end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{c} \varepsilon^{\mu\nu\sigma} p_\nu p^2 E(p^2) = 0 \Rightarrow E(p^2) = 0; \\ \varepsilon^{\mu\nu\rho} p_\nu p_\rho p_\sigma E(p^2) + (\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}) p_\nu p_\tau F(p^2) = \frac{4\pi}{k}[\delta_{\mu\sigma} - \frac{p_\mu p_\sigma}{p^2}] \end{array} \right. \Rightarrow$$

$$\Rightarrow (\delta_{\mu\sigma} p^2 - p_\mu p_\sigma) F(p^2) = \frac{4\pi}{k}[\delta_{\mu\sigma} - \frac{p_\mu p_\sigma}{p^2}] \Rightarrow F(p^2) = \frac{4\pi}{k}\frac{1}{p^2}\,.$$

To pass from the first to the second row of this list of algebraic steps we use the well-known result $\varepsilon^{\rho\mu\nu}\varepsilon_{\rho\sigma\tau} = (\delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma})$. Replacing the results of the system (2.32) in the formal expressions (2.32) we obtain

$$\begin{aligned}
\left\langle A^a_\rho(x) A^b_\sigma(y) \right\rangle &= \delta^{ab} \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)} \varepsilon_{\rho\sigma\tau} p_\tau \frac{4\pi}{k}\frac{1}{p^2} = \\
&= \frac{4\pi}{k}\delta^{ab} \int \frac{d^3p}{(2\pi)^3} \varepsilon_{\rho\sigma\tau}(-i)\partial_\tau e^{ip\cdot(x-y)} \frac{1}{p^2} = \\
&= -i\,\frac{4\pi}{k}\delta^{ab}\varepsilon_{\rho\sigma\tau}\partial_\tau \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)} \frac{1}{p^2}\,; \qquad (2.33)
\end{aligned}$$

$$\left\langle A^a_\rho(x) B^b(y) \right\rangle = i\frac{4\pi}{k}\delta^{ab}\partial_\rho \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)} \frac{1}{p^2}\,; \qquad (2.34)$$

$$\left\langle B^a(x) A^b_\sigma(y) \right\rangle = -i\frac{4\pi}{k}\delta^{ab}\partial_\sigma \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)} \frac{1}{p^2}\,; \qquad (2.35)$$

$$\left\langle B^a(x) B^b(y) \right\rangle = 0\,. \qquad (2.36)$$

Thus, to write all the propagators in the coordinate representation, we have to find the Fourier anti-transformation of the term $\frac{1}{p^2}$. This is an easy task, in fact it is proportional to the Green function of the three-dimensional Laplacian $\Delta = \partial^\mu\partial_\mu$, in the momentum representation, in fact

$$\Delta G(x-y) = \delta(x-y) \Rightarrow -p^2 G(p) = 1 \Rightarrow G(p) = -\frac{1}{p^2}\,. \qquad (2.37)$$

This equation is well-known, for example the gravitational and the electric potentials fulfil it and its solution is (in our three dimensional case)

$$G(x-y) = -\frac{1}{4\pi}\frac{1}{|x-y|} \left(= \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)}(-)\frac{1}{p^2} \right)\,.$$

Performing the partial derivative which appears in the propagators expression we find

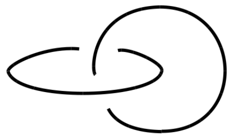
$$\partial_i G(x-y) = \frac{1}{4\pi}\frac{(x-y)_i}{|x-y|^3} \, ,$$

that inserted in equation (2.33) gives the propagators for the bosonic field (2.14-2.16).

It is not difficult to show that no more computation are needed to find the ghost propagator noticing that integrating by parts in (2.13) the fermionic quadratic part is merely the Laplacian.

2.7.2 Linking number

To compute the integral (2.8) it is convenient to consider a Seifert surface Σ of the knot K_1, *i.e.* $\partial\Sigma = K_1$ and use the Stokes theorem. Moreover, to simplify the calculation in favour of a clearer illustration of the integral features, we can take C_1 and C_2 to be the components of the *Hopf link*.



The stokes theorem states that given a knot K and a $\mathcal{C}^1$ one-form V

$$\oint_K V = \int_\Sigma \mathrm{d}V,$$

where d is the external differential and Σ is a Seifert surface of the knot K. In expression (2.8), for fixed y and ν, we can consider the one-form

$$V^{\mu\nu}\mathrm{d}x^\mu = \varepsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{|x-y|^3}\mathrm{d}x^\mu = -\varepsilon_{\mu\nu\rho}\partial_x^\rho\frac{1}{|x-y|}\mathrm{d}x^\mu,$$

whose differential is

$$\begin{aligned}\mathrm{d}(V^{\mu\nu}\mathrm{d}x^\mu) &= \partial_\tau^x V^{\mu\nu}\mathrm{d}x^\tau\wedge\mathrm{d}x^\mu = \varepsilon_{\alpha\tau\mu}\partial_\tau^x V^{\mu\nu}\mathrm{d}\sigma^\alpha = -\varepsilon_{\alpha\tau\mu}\partial_\tau^x\varepsilon_{\mu\nu\rho}\partial_x^\rho\frac{1}{|x-y|}\mathrm{d}\sigma^\alpha = \\ &= -(\delta_\nu^\alpha\delta_\rho^\tau - \delta_\rho^\alpha\delta_\nu^\tau)\partial_\tau^x\partial_x^\rho\frac{1}{|x-y|}\mathrm{d}\sigma^\alpha = (\partial_\nu^x\partial_x^\alpha - \delta_\nu^\alpha\partial_x^2)\frac{1}{|x-y|}\mathrm{d}\sigma^\alpha,\end{aligned}$$

where $\mathrm{d}\sigma^\alpha$ is the one-form dual the two-form $\mathrm{d}x^\tau\wedge\mathrm{d}x^\mu$ in regard to the *Hodge product*. Namely, fixed Σ the Seifert surface of K_1, $\mathrm{d}\sigma^\alpha$ in each $x\in\Sigma$ is normal to Σ and has modulo equal to the infinitesimal surface portion. The one-form V^ν is $\mathcal{C}^1$ for all y except for $y = x$. Therefore, by introducing the parametrization $K_2 =$

$\{y^\nu(t), t \in [0,1]$ such that $y^\nu(0) = y^\nu(1) \in \Sigma\}$ and by using the Stokes theorem, we can rewrite expression (2.8) as the following limit

$$\begin{aligned}
\varphi(K_1,K_2) &= \lim_{\chi\to 0}\frac{1}{4\pi}\int_{0+\chi}^{1-\chi}\mathrm{d}t\,\dot{y}^\nu(t)\int_\Sigma \mathrm{d}\sigma^\alpha(\partial^x_\nu\partial^\alpha_x - \delta^\alpha_\nu\partial^2_x)\frac{1}{|x-y(t)|} = \\
&= \frac{1}{4\pi}\lim_{\chi\to 0}\left[\int_{0+\chi}^{1-\chi}\mathrm{d}t\,\dot{y}^\nu(t)\,\partial^y_\nu\int_\Sigma \mathrm{d}\sigma^\alpha\frac{(x-y(t))_\alpha}{|x-y(t)|^3} - \right.\\
&\quad\left. - \int_{0+\chi}^{1-\chi}\mathrm{d}t\,\dot{y}^\nu(t)\int_\Sigma \mathrm{d}\sigma^\nu\delta^3(x-y(t))\right] = \\
&= \frac{1}{4\pi}\lim_{\chi\to 0}\left[\int_\Sigma \mathrm{d}\sigma^\alpha\frac{(x-y(1-\chi))_\alpha}{|x-y(1-\chi)|^3} - \int_\Sigma \mathrm{d}\sigma^\alpha\frac{(x-y(0+\chi))_\alpha}{|x-y(0+\chi)|^3}\right] = \\
&= \frac{1}{4\pi}\lim_{\chi\to 0}\left[\int_\Sigma \frac{\mathrm{d}\sigma^\alpha}{|r(\chi)|^2}\hat{r}_\alpha(\chi) - \cdots\right] = \frac{1}{4\pi}\lim_{\chi\to 0}\left[\int_\Sigma \mathrm{d}\Omega(\chi) - \cdots\right] = \\
&= \frac{1}{4\pi}\lim_{\chi\to 0}\,[2\pi - (-)2\pi] = 1\,,
\end{aligned} \tag{2.38}$$

where $r^\alpha(\chi) = (x - y(\chi))^\alpha$ and $\mathrm{d}\Omega(\chi)$ is the infinitesimal solid angle for fixed χ. Even if one has to introduce a metric to associate $\mathrm{d}x^\tau \wedge \mathrm{d}x^\mu$ to its Hodge dual, the final result does not depend on it.

Chapter 3

Three-dimensional torus

In this work we consider the three dimensional torus $\mathbb{T}^3 = S^1 \times S^1 \times S^1$ and from now on let $G = SU(2)$ be the structure group. First, we fix the notation representing $\mathbb{T}^3$ as a cube with opposite faces identified. It is characterized by three coordinate orthogonal axes x^μ. Since the theory does not depend on the metric, there exist an arbitrariness in the choice of the length of each axis. In our notation, each coordinate has to be thought as an angle. We set $x^\mu \in [-\pi, \pi]$ for each $\mu = 1, 2, 3$. An orthonormal complete set in the chosen parametrization, $\mathbb{T}^3 = [-\pi, \pi] \times [-\pi, \pi] \times [-\pi, \pi]$, of the Hilbert space of periodic functions on this manifold is

$$\frac{1}{(\sqrt{2\pi})^3} e^{ip_\gamma x^\gamma} \text{ with } p_\gamma = 0, \pm 1, \pm 2, ... \text{ for } \gamma = 1, 2, 3. \tag{3.1}$$

We recall that in the chosen basis the three-dimensional Dirac delta function can be written as

$$\delta^3(x - y) = \frac{1}{(2\pi)^3} \sum_{p_\sigma} e^{ip_\sigma (x-y)^\sigma}. \tag{3.2}$$

Since the gauge group is isomorphic to S^3, its first nontrivial homotopy group is the third one. It follows that in any intersection between two open three-balls, the gauge transition function can be reduced to the identity. Therefore, the principal bundle is trivial and we can take the connection A to be described by a one-form defined globally on the whole manifold.

3.1 Flat connections

The solutions of the equations of motion in a manifold with a non trivial topology are not merely the pure gauge connections as in $\mathbb{R}^3$. In particular [9]-[12], gauge inequivalent solutions come up in direct relation to the first homotopy group, $\pi_1(M)$. In order to clearly describe how these solutions arise from this topological features of the manifold, let us rewrite the holonomy expression (2.18) of the connection A

on a closed path γ, *i.e.* on the loop γ:

$$h(\gamma; A) = \mathrm{P}\, e^{i \int_\gamma \mathrm{d}x^\mu A_\mu(x)} \,. \tag{3.3}$$

Since the variation of this expression under continuous deformations of the path γ is proportional to the curvature of A_μ, when the latter is flat, the holonomy depends only on the homotopy class of the loop. Thus, $h(\gamma)$ defines a representation of $\pi_1(M)$ in the structure group G. Furthermore, the expression (3.3) is covariant under gauge transformations of the connection, therefore one has the following relation:

$$\frac{\{\textit{flat connections}\}}{\{\textit{gauge transformations}\}} \cong \frac{\{\pi_1(M) \textit{ representations in the gauge group }\}}{\{\textit{gauge group adjoint action}\}}$$

In the considered manifold, the first homotopy group $\pi_1(\mathbb{T}^3) = \mathbb{Z}^3$ is Abelian and is generated by three loops, let us call them β^μ, that can be parametrized as follows:

$$\beta^1 = (t_1, 0, 0);$$
$$\beta^2 = (0, t_2, 0);$$
$$\beta^3 = (0, 0, t_3);$$

with $t_i \in [-\pi, \pi]$ for each $i = 1, 2, 3$. Therefore, all gauge potentials with zero curvature can be written as follows

$$A_\mu(x) = U(x)^{-1} \alpha_\mu U(x) - i U(x)^{-1} \partial_\mu U(x), \tag{3.4}$$

where $U(x)$ is an single-valued gauge transformation, while α_μ is a vector in the algebra and has to fulfil the conditions:

$$[\alpha_\mu, \alpha_\nu] = 0 \quad \text{and} \quad \alpha_\mu = \text{constant} \quad \forall\, \mu, \nu = 1, 2, 3.$$

We fix a basis of the algebra of $SU(2)$ in the fundamental representation such that $T^a = \frac{\sigma^a}{2}$ with $a = 1, 2, 3$, where σ^a are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We can take α parallel to one of these generators, since it is Abelian and defined up to constant gauge transformations, which can be reabsorbed by $U(x)$. For example, we can choose $\alpha_\mu = \alpha^3_\mu T^3$. Moreover, if we define $[\gamma] = n^\mu$ the homotopy class of γ, where $n^\mu \in \mathbb{Z}\ \forall \mu$ stand for the powers of the loops generators, we can write:

$$h(\gamma; \alpha) = e^{i\, 2\pi\, n^\mu \alpha_\mu} = \cos(\pi\, n^\mu \alpha^3_\mu)\mathbb{I} + 2\, i \sin(\pi\, n^\mu \alpha^3_\mu) T^3, \tag{3.5}$$

therefore, due to the periodicity of the cosine and the sine, two flat connections which differ from each other only up to a term $\alpha'_\mu - \alpha_\mu = 2m_\mu$ with $m^3_\mu \in \mathbb{Z}^3$, lead to the same representation of the $\pi_1(M)$.

Thus, we can summarize

$$[\alpha_\mu, \alpha_\nu] = 0\,,\ \alpha_\mu = \text{constant and } -1 < \alpha^3_\mu < 1 \quad \forall\, \mu, \nu = 1, 2, 3.$$

3.2 Functional integration

The operator in the quadratic part of the Chern-Simons action is purely differential, thus it annihilates the constant fields. In the standard $\mathbb{R}^3$ case the constant fields can not be normalized; by appropriate boundary conditions, constant fields are excluded from the functional integration. In the manifold $\mathbb{T}^3$, instead, we have to define the functional integral also on these degrees of freedom, but it is not obvious how to perform the pertubative expansion since the constant fields do not propagate. We can even try to perform the BRST method with different kinds of gauge fermions in order to obtain a quadratic operator which does not annihilate the constant, but the zero modes, by definition, cannot be fixed by the BRST procedure. We try to bypass these complication by introducing a background *constant flat* connection α_μ. Since we assume that the path-integral is invariant under translation in the fields variable, we can write:

$$\int_{\mathbb{T}^3} \mathrm{D}[A+\alpha]\ e^{iS[A+\alpha]} = \int_{\mathbb{T}^3} \mathrm{D}[A]\ e^{iS[A+\alpha]} = \int_{\mathbb{T}^3} \mathrm{D}[A]\ e^{iS'[A]}. \tag{3.6}$$

The primed action in the above expression can be obtained from the Chern-Simons action by means of the substitution $\partial_\mu \cdot \to \mathrm{D}_\mu(\alpha) = \partial_\mu \cdot + i[\alpha_\mu, \cdot]$. In fact,

$$\begin{aligned} S'[A] &= S[A+\alpha] = S[\alpha] + S[A] + \\ &\quad + \frac{k}{4\pi}\int_{\mathbb{T}^3} d^3x\, \varepsilon^{\mu\nu\rho}\mathrm{tr}\,[\, A_\mu\partial_\nu\alpha_\rho + \alpha_\mu\partial_\nu A_\rho + 2i\ A_\mu\alpha_\nu\alpha_\rho + 2i\ A_\mu\alpha_\nu A_\rho\,] = \\ &= S[A] + \frac{k}{4\pi}\int_{\mathbb{T}^3} d^3x\, \varepsilon^{\mu\nu\rho}\mathrm{tr}\,[\, A_\mu\partial_\nu\alpha_\rho - A_\mu\partial_\rho\alpha_\nu + i\ A_\mu[\alpha_\nu,\alpha_\rho] + iA_\mu[\alpha_\nu, A_\rho]] = \\ &= \frac{k}{4\pi}\int_{\mathbb{T}^3} d^3x\, \varepsilon^{\mu\nu\rho}\mathrm{tr}\left[\, A_\mu D_\nu(\alpha)A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho\right] + \frac{k}{4\pi}\int_{\mathbb{T}^3} d^3x\, \varepsilon^{\mu\nu\rho}\mathrm{tr}\,[A_\mu F_{\nu\rho}(\alpha)] = \\ &= \frac{k}{4\pi}\int_{\mathbb{T}^3} d^3x\, \varepsilon^{\mu\nu\rho}\mathrm{tr}\left[\, A_\mu D_\nu(\alpha)A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho\right]. \end{aligned} \tag{3.7}$$

We want to underline that in the second step of the above computation, we integrate by part and no boundary term comes up because $\mathbb{T}^3$ as no boundary.

The advantage which we get with these new definitions is that, with $\alpha_\mu \neq 0$, the kernel of $\mathrm{D}_\mu(\alpha)$ is three-dimensional, one dimension for each spatial direction.Let us recall that the space of the zero modes of the ordinary derivative operator ∂_μ is nine-dimensional in $\mathbb{T}^3$. In fact,

$$D_\mu(\alpha)\lambda_\nu = 0 \Rightarrow \lambda_\nu \propto \alpha_\nu.$$

The key idea is that we perform the computation of the expectation values by means of the path-integral with the action $S'[A] = S[A+\alpha]$. the function of the field with the exponential of S. Since we are interested in the perturbative approach, it is convenient to separate the kernel of the covariant derivative $\mathrm{D}_\mu(\alpha)$ from the rest of A:

$$A = \lambda + \tilde{A}; \quad \Rightarrow \mathrm{D}A = \mathrm{D}\lambda\, \mathrm{D}\tilde{A}, \tag{3.8}$$

where the functional differential $\mathrm{D}A$ factorizes as $\mathrm{D}\lambda\, \mathrm{D}\tilde{A}$, because λ and ω concern independent degrees of freedom. By means of this separation, we compute the expectation values, initially, by carrying out the integration only in the ω degrees of freedom, in order to obtain an expression of the propagator. The results that we obtain have to be understood as dependent on λ and, at the end, we perform the integration in this variables.

The Chern-Simons action S[A] is invariant under gauge transformations of the following form

$$A'_\mu(x) = g^{-1}(x)A_\mu(x)g(x) - ig^{-1}(x)\partial_\mu g(x), \tag{3.9}$$

with $g(x)$ single-valued on $\mathbb{T}^3$. Since $S'[A] = S[A+\alpha]$, the new gauge transformations are:

$$A'_\mu(x) = g^{-1}(x)[\alpha_\mu, g(x)] + g^{-1}(x)A_\mu(x)g(x) - ig^{-1}(x)\partial_\mu g(x),$$

which infinitesimally give the variation

$$\delta A = \mathrm{D}(\alpha)\varepsilon + i[A, \theta]. \tag{3.10}$$

Let us underline that, given the flat connection α, we can define the *covariant external differential* as a generalization of the external differential d, by substituting the partial derivative with the covariant derivative with respect to the fixed connection. Therefore, the action of the covariant external differential on a general one-form f, with values in the algebra, can be written as follows

$$\mathrm{d}_\alpha f = \mathrm{D}_\mu(\alpha) f \mathrm{d}x^\mu = (\partial_\mu f + i[\alpha_\mu, f])\mathrm{d}x^\mu. \tag{3.11}$$

Since the connection α is flat, d_α is nilpotent, in fact

$$\begin{aligned}
\mathrm{d}_\alpha\left(\mathrm{d}_\alpha f\right) &= \mathrm{d}_\alpha(\mathrm{D}_\mu f \mathrm{d}x^\mu) = \mathrm{D}_\nu(\mathrm{D}_\mu f)\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = \\
&= (\partial_\nu\partial_\mu f + i\partial_\nu[\alpha_\mu, f] + i[\alpha_\nu, \partial_\mu f] - [\alpha_\nu, [\alpha_\mu, f]])\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = \\
&= \left(i[\partial_\nu\alpha_\mu, f]] - \frac{1}{2}[[\alpha_\nu, \alpha_\mu], f]\right)\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = \\
&= i\frac{1}{2}[F_{\nu\mu}(\alpha), f]\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = 0.
\end{aligned} \tag{3.12}$$

3.2.1 BRST

Following the BRST method we introduce the ghost and the antighost fields, respectively C and $\bar{C}$, and a bosonic field B and define the BRST transformations from the variations (3.10):

$$s\,A_\mu = D_\mu(\alpha)C + i[A_\mu, C]; \quad s\,\bar{C} = B;$$

$$s\,C = \tfrac{1}{2}[C, C]; \qquad s\,B = 0;$$

The feature (3.12) of d_α is essential for the BRST procedure for what follows. Since d_α is nilpotent at the second order, we can define its cohomology as the space of the *colsed form* up to the *exact form* with respect to the action of d_α, analogously to the *de Rham cohomology*. The quadratic part which appear in S' can be written as

$$A \wedge \mathrm{d}_\alpha A,$$

therefore, it is singular on closed one-forms. In order to fix the gauge degrees of freedom, the gauge fermion has to be chose in such a way that the function $G(A)$ in the expression (1.25) fulfils the following characteristic:

$$G(\mathrm{d}_\alpha \theta) = 0 \Rightarrow \theta = 0. \tag{3.13}$$

Roughly speaking, the above demand neglects the exact part of A. We can naïvely explain this feature by recalling that the action term, in the boson sector, deduced by the proposition (3.13) is a sort of constraint with B acting as the Lagrangian multiplier. Strictly speaking, the above cited action term added to the original action corrects the quadratic operator. By inverting the direction of the implication in proposition (3.13), we get that this correction does not annihilate the gauge degrees of freedom. Moreover, since it is a BRST variation, the physical expectation values are not affected by this modification of the action. A suitable choice in this case is the background gauge fermion:

$$\Psi = -\frac{k}{2\pi}\int_{\mathbb{T}^3} \mathrm{d}^3x \,\mathrm{tr}\left[\bar{C} D_\mu(\alpha) A_\mu\right]$$

$$\begin{aligned}
\Rightarrow s\Psi \;&=\; -\tfrac{k}{2\pi}\textstyle\int_{\mathbb{T}^3} \mathrm{d}^3x \,\mathrm{tr}[B\mathrm{D}_\mu(\alpha)A_\mu - \bar{C}\mathrm{D}_\mu(\alpha)\mathrm{D}_\mu(\alpha + A)C] \\
&=\; \tfrac{k}{4\pi}\textstyle\int_{\mathbb{T}^3} \mathrm{d}^3x \,\mathrm{tr}[A_\mu \mathrm{D}_\mu(\alpha)B - B\mathrm{D}_\mu(\alpha)A_\mu] + \\
&\quad + \tfrac{k}{4\pi}\textstyle\int_{\mathbb{T}^3} \mathrm{d}^3x \,\mathrm{tr}\left[2\bar{C}\mathrm{D}_\mu(\alpha)\mathrm{D}_\mu(\alpha)C + i2\bar{C}\mathrm{D}_\mu(\alpha)[A_\mu, C]\right].
\end{aligned}$$

Now we can write the total action, which satisfies the BRST theory conditions, *i.e.* $S'_{tot} = S' + \delta\Psi$. In the following we write the total action divided in its quadratic parts and the interaction term. Moreover, to write compactly the boson quadratic term we define the vector field $\Omega^a_i = (A^a_\mu, B^a); i = \mu, 4$ and we call $D^{ab}_\mu(\alpha) = \delta^{ab}\partial_\mu - \varepsilon^{a3b}\alpha^3_\mu$ the covariant derivative with respect to α in the adjoint representation.

$$S'_{tot} = S'_{\Omega\Omega} + S'_{\bar{C}C} + S'_I$$

$$\begin{aligned}
S'_{\Omega\Omega} \;&=\; \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x \,\frac{k}{4\pi}\left[A^a_\mu \varepsilon^{\mu\nu\rho} D^{ab}_\nu(\alpha) A^b_\rho + A^a_\mu D^{ab}_\mu(\alpha) B^b - B^a D^{ab}_\rho(\alpha) A^b_\rho\right] = \\
&=\; \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x \,\Omega^a_i \frac{k}{4\pi}\begin{pmatrix} \varepsilon^{\mu\nu\rho} D^{ab}_\nu(\alpha) & -D^{ab}_\mu(\alpha) \\ D^{ab}_\rho(\alpha) & 0 \end{pmatrix}_{ij} \Omega^b_j,
\end{aligned} \tag{3.14}$$

where we set $i = \mu, 4; j = \rho, 4;$

$$S'_{\bar{C}C} \;=\; \int_{\mathbb{T}^3} \mathrm{d}^3x\, \bar{C}^a \frac{k}{4\pi} \left[\mathrm{D}^{ac}_\mu(\alpha)\mathrm{D}^{cb}_\mu(\alpha)\right] C^b \tag{3.15}$$

$$S'_I \;=\; \int_{\mathbb{T}^3} \mathrm{d}^3x\, \frac{k}{4\pi} \left[-\frac{1}{3!}\varepsilon^{\mu\nu\rho}\varepsilon_{abc}\, A^a_\mu A^b_\nu A^c_\rho - \varepsilon_{acd}(\mathrm{D}^{ab}_\mu(\alpha)\bar{C}^b)A^c_\mu C^d\right] \tag{3.16}$$

3.3 Conserved charge

By fixing the constant part of the connection, we globally break the $SU(2)$ symmetry. Nevertheless, the action is invariant under the transformations of a residual $U(1)$ global symmetry which corresponds to the rotation generated by T^3. It is convenient to define proper combinations of the fields which diagonalize the propagator and make manifest this symmetry. Therefore, we define the following unitary matrix

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \qquad V^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & +i \end{pmatrix} = V^{-1}. \tag{3.17}$$

The above matrices, acting on the orthogonal sector with respect to T^3, give the following combination of the connection components

$$\psi_\mu = \begin{pmatrix} A^+_\mu \\ A^-_\mu \end{pmatrix} = V\begin{pmatrix} A^1_\mu \\ A^2_\mu \end{pmatrix}, \qquad \bar{\psi}_\mu = \begin{pmatrix} A^-_\mu, & A^+_\mu \end{pmatrix} = \begin{pmatrix} A^1_\mu, & A^2_\mu \end{pmatrix} V^\dagger = (\psi^t_\mu)^*.$$

Indeed, we apply this transformations also to the auxiliary field B

$$\eta = \begin{pmatrix} B^+ \\ B^- \end{pmatrix} = V\begin{pmatrix} B^1 \\ B^2 \end{pmatrix}, \qquad \bar{\eta} = \begin{pmatrix} B^-, & B^+ \end{pmatrix} = \begin{pmatrix} B^1, & B^2 \end{pmatrix} V^\dagger,$$

to the ghost, and to the antighost

$$\chi = \begin{pmatrix} C^+ \\ C^- \end{pmatrix} = V\begin{pmatrix} C^1 \\ C^2 \end{pmatrix}, \qquad \bar{\chi} = \begin{pmatrix} \bar{C}^-, & \bar{C}^+ \end{pmatrix} = \begin{pmatrix} \bar{C}^1, & \bar{C}^2 \end{pmatrix} V^\dagger.$$

In order to stress the meaning of the charge analogy, from now on we use the index 0 for the fields component parallel to T^3. Since we want to rewrite the action in the new fields, it is convenient to compute the adjoint action of V on the covariant derivative on the orthogonal sector:

$$\tilde{\partial_\mu} = V\mathrm{D}_\mu V^\dagger = \begin{pmatrix} \partial_\mu - i\alpha_\mu & 0 \\ 0 & \partial_\mu + i\alpha_\mu \end{pmatrix} \tag{3.18}$$

In the new basis, by defining $\Omega_i^0 = (A_\mu^0, B^0)$, $\Theta_i = (\psi_\mu, \eta)$ and $\bar{\Theta}_i = \left(\bar{\psi}_\mu, \bar{\eta}\right)$,we can rewrite the expressions (3.14)-(3.16) as follows

$$S'_{\Omega^0\Omega^0} = \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x\, \Omega_i^0 \frac{k}{4\pi} \begin{pmatrix} \varepsilon^{\mu\nu\rho}\partial_\nu & -\partial_\mu \\ \partial_\rho & 0 \end{pmatrix}_{ij} \Omega_j^0 = \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x\, \Omega^0\, \mathbf{\Gamma}\, \Omega^0 \quad (3.19)$$

$$S'_{\bar{\Theta}\Theta} = \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x\, \bar{\Theta}_i \frac{k}{4\pi} \begin{pmatrix} \varepsilon^{\mu\nu\rho}\tilde{\partial}_\nu & -\tilde{\partial}_\mu \\ \tilde{\partial}_\rho & 0 \end{pmatrix}_{ij} \Theta_j = \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}^3x\, \bar{\Theta}\, \tilde{\mathbf{\Gamma}}\, \Theta \quad (3.20)$$

$$S'_{\bar{C}^0C^0} = \int_{\mathbb{T}^3} \mathrm{d}^3x\, \bar{C}^0 \frac{k}{4\pi}\left[\partial_\mu\partial_\mu\right] C^0 = \int_{\mathbb{T}^3} \mathrm{d}^3x\, \bar{C}^0\, \mathbf{\Lambda}\, C^0 \quad (3.21)$$

$$S'_{\bar{\chi}\chi} = \int_{\mathbb{T}^3} \mathrm{d}^3x\, \bar{\chi} \frac{k}{4\pi}\left[\tilde{\partial}_\mu\tilde{\partial}_\mu\right] \chi = \int_{\mathbb{T}^3} \mathrm{d}^3x\, \bar{\chi}\, \tilde{\mathbf{\Lambda}}\, \chi \quad (3.22)$$

$$\begin{aligned} S'_I = & \int_{\mathbb{T}^3} \mathrm{d}^3x\, \frac{k}{4\pi}\left\{i\varepsilon^{\mu\nu\rho}A_\mu^+ A_\nu^0 A_\rho^- - i\partial_\mu \bar{C}^0\left(A_\mu^- C^+ - A_\mu^+ C^-\right) -\right. \\ & -iA_\mu^0\left[\left\{(\partial_\mu + i\alpha_\mu)\bar{C}^-\right\}C^+ - \left\{(\partial_\mu - i\alpha_\mu)\bar{C}^+\right\}C^-\right] - \\ & \left. -i\left[\left\{(\partial_\mu - i\alpha_\mu)\bar{C}^+\right\}A_\mu^- - \left\{(\partial_\mu + i\alpha_\mu)\bar{C}^-\right\}A_\mu^+\right]C^0\right\} = \\ = & \int_{\mathbb{T}^3} \mathrm{d}^3x\, \frac{k}{4\pi}\left\{\frac{i}{2}A_\mu^0\varepsilon^{\mu\nu\rho}\bar{\psi}_\nu\psi_\rho - iA_\mu^0(\tilde{\partial}_\mu\bar{\chi})\chi\right. \\ & \left. -i(\partial_\mu\bar{C}^0)\bar{\psi}_\mu\chi + i(\tilde{\partial}_\mu\bar{\chi})\psi_\mu C^0\right\}. \end{aligned} \quad (3.23)$$

Since the quadratic part of the action factorizes according to the breaking pattern of the global symmetry, it is suitable to improve the separation prescription (3.8). In fact, the residual zero mode λ is contained only in the A^0 sector. Moreover, also the BRST fields contain zero modes which prevent the computation of the propagators. Therefore, we the following definitions:

$$A^0 \to A^0 + \lambda\,, \qquad \mathrm{D}A = \mathrm{D}\lambda\mathrm{D}A^0\mathrm{D}A^+\mathrm{D}A^- = \mathrm{D}\lambda\mathrm{D}\tilde{A}; \quad (3.24)$$
$$B^0 \to B^0 + \beta\,, \qquad \mathrm{D}B = \mathrm{D}\beta\mathrm{D}B^0\mathrm{D}B^+\mathrm{D}B^- = \mathrm{D}\beta\mathrm{D}\tilde{B}; \quad (3.25)$$
$$C^0 \to C^0 + \zeta\,, \qquad \mathrm{D}C = \mathrm{D}\zeta\mathrm{D}C^0\mathrm{D}C^+\mathrm{D}C^- = \mathrm{D}\zeta\mathrm{D}\tilde{C}; \quad (3.26)$$
$$\bar{C}^0 \to \bar{C}^0 + \bar{\zeta}\,, \qquad \mathrm{D}\bar{C} = \mathrm{D}\bar{\zeta}\mathrm{D}\bar{C}^0\mathrm{D}\bar{C}^+\mathrm{D}\bar{C}^- = \mathrm{D}\bar{\zeta}\mathrm{D}\tilde{\bar{C}}; \quad (3.27)$$

where λ, β, ζ and $\bar{\zeta}$ are, respectively, the non-propagating component for each field.

All these variables appear only in the interaction terms; therefore, it is possible to find the expression of the propagators for the rest of the field variables.

3.3.1 Propagator

Since we are in a compact manifold, the propagators in the momentum representation can be expressed by their coefficient with respect to their Fourier series. Called p_μ the momentum conjugated to x^μ, the partial derivative ∂_μ correspond to the multiplicative operator ip_μ. It is favourable to define also

$$i\tilde{p} = i\begin{pmatrix} p_\mu - \alpha_\mu & 0 \\ 0 & p_\mu + \alpha_\mu \end{pmatrix}$$

related to the covariant derivative. Therefore, we can write:

$$\mathbf{\Gamma}(p) = \frac{ik}{4\pi}\begin{pmatrix} \varepsilon^{\mu\nu\rho}p_\nu & -p_\mu \\ p_\rho & 0 \end{pmatrix}, \quad \mathbf{\Lambda}(p) = -\frac{k}{4\pi}p_\mu p_\mu = -\frac{k}{4\pi}p^2; \tag{3.28}$$

$$\tilde{\mathbf{\Gamma}}(p) = \frac{ik}{4\pi}\begin{pmatrix} \varepsilon^{\mu\nu\rho}\tilde{p}_\nu & -\tilde{p}_\mu \\ \tilde{p}_\rho & 0 \end{pmatrix}, \quad \tilde{\mathbf{\Lambda}}(p) = -\frac{k}{4\pi}\tilde{p}_\mu\tilde{p}_\mu. \tag{3.29}$$

In order to find the inverse of the matrix $\mathbf{\Gamma}$ for each p, it turns out to be useful to write it as

$$\mathbf{\Gamma}^{-1} = \mathbf{\Gamma}(\mathbf{\Gamma}^2)^{-1}.$$

In fact, $(\mathbf{\Gamma}^2)^{-1}$ is easier to calculate because is the inverse of $\mathbf{\Gamma}$ squared, which is block-diagonal and all the blocks are mutually equal and proportional to $\mathbf{\Lambda}$

$$\mathbf{\Gamma}^2 = \frac{k}{4\pi}\begin{pmatrix} \mathbf{\Lambda} & 0 & 0 & 0 \\ 0 & \mathbf{\Lambda} & 0 & 0 \\ 0 & 0 & \mathbf{\Lambda} & 0 \\ 0 & 0 & 0 & \mathbf{\Lambda} \end{pmatrix} = \frac{k}{4\pi}\mathbf{\Lambda}\,\mathbb{I}_{4\times 4}, \tag{3.30}$$

$$\Rightarrow \mathbf{\Gamma}^{-1} = \frac{4\pi}{k}\mathbf{\Gamma}[\mathbf{\Lambda}^{-1}\,\mathbb{I}_{4\times 4}]. \tag{3.31}$$

Indeed, this is true also for $\tilde{\mathbf{\Gamma}}$. Thus, it remains only to calculate the inverse of the ghost quadratic operator.

Since we leave out the constant parts in the components parallel to T^3, in this case the equation to solve, in order to find the Green function G for the generic operator $\mathbf{O}$, is

$$\mathbf{O}(x)G(x-y) = \delta^3(x-y) - 1 = \frac{1}{(2\pi)^3}\sum_{p_\sigma \neq 0} e^{ip_\sigma(x-y)^\sigma}.$$

Therefore, in the chosen complete set (3.1), we obtain

$$\begin{aligned}
\left\langle C^0(x)\bar{C}^0(y)\right\rangle &= \mathbf{\Lambda}^{-1}(x-y) = \frac{4\pi}{k}\sum_{p_\gamma\neq 0} -\frac{1}{p^2}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}, && (3.32)\\
\left\langle \Omega^0(x)\Omega^0(y)\right\rangle &= \mathbf{\Gamma}^{-1}(x-y) = \frac{4\pi}{k}\sum_{p_\gamma\neq 0} -i\begin{pmatrix} \varepsilon^{\mu\nu\rho}\frac{p_\nu}{p^2} & -\frac{p_\mu}{p^2} \\ \frac{p_\rho}{p^2} & 0\end{pmatrix}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}. && (3.33)
\end{aligned}$$

Whereas, in regard to the components orthogonal to T^3, since

$$\tilde{p}_\mu\tilde{p}_\mu = \begin{pmatrix} (p-\alpha)^2 & 0 \\ 0 & (p+\alpha)^2\end{pmatrix}$$

is invertible for $0<\alpha<1$, we obtain

$$\left\langle \chi(x)\bar{\chi}(y)\right\rangle = \frac{4\pi}{k}\sum_{p_\gamma} -\begin{pmatrix} \frac{1}{(p-\alpha)^2} & 0 \\ 0 & \frac{1}{(p+\alpha)^2}\end{pmatrix}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}, \tag{3.34}$$

$$\left\langle \psi_\mu(x)\bar{\psi}_\rho(y)\right\rangle = \frac{4\pi}{k}\sum_{p_\gamma} -i\varepsilon^{\mu\nu\rho}\begin{pmatrix} \frac{(p-\alpha)_\nu}{(p-\alpha)^2} & 0 \\ 0 & \frac{(p+\alpha)_\nu}{(p+\alpha)^2}\end{pmatrix}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}, \tag{3.35}$$

$$\left\langle \eta(x)\bar{\psi}_\mu(y)\right\rangle = \frac{4\pi}{k}\sum_{p_\gamma} -i\begin{pmatrix} \frac{(p-\alpha)_\mu}{(p-\alpha)^2} & 0 \\ 0 & \frac{(p+\alpha)_\mu}{(p+\alpha)^2}\end{pmatrix}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}, \tag{3.36}$$

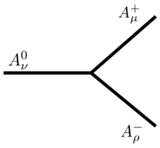
$$\left\langle \psi_\rho(x)\bar{\eta}(y)\right\rangle = \frac{4\pi}{k}\sum_{p_\gamma} i\begin{pmatrix} \frac{(p-\alpha)_\rho}{(p-\alpha)^2} & 0 \\ 0 & \frac{(p+\alpha)_\rho}{(p+\alpha)^2}\end{pmatrix}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}. \tag{3.37}$$

Since the interaction term of the total action does not contain the field B, the latter does not appear in any term of the perturbative expansion. Thus, we are interested only in the propagators of the other fields. Finally, for the "charged" fields we can write the following expressions of the propagators which we use in the further computation:

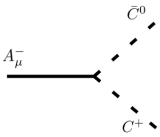
$$\begin{aligned}
\left\langle C^0(x)\bar{C}^0(y)\right\rangle &= -\frac{4\pi}{k}\sum_{p_\gamma\neq 0}\frac{1}{p^2}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}; && (3.38)\\
\left\langle C^\mp(x)\bar{C}^\pm(y)\right\rangle &= -\frac{4\pi}{k}\sum_{p_\gamma}\frac{1}{(p\mp\alpha)^2}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}; && (3.39)\\
\left\langle A^0_\mu(x)A^0_\rho(y)\right\rangle &= -i\frac{4\pi}{k}\sum_{p_\gamma\neq 0}\varepsilon^{\mu\nu\rho}\frac{p_\nu}{p^2}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}; && (3.40)\\
\left\langle A^\pm_\mu(x)A^\mp_\rho(y)\right\rangle &= -i\frac{4\pi}{k}\sum_{p_\gamma}\varepsilon^{\mu\nu\rho}\frac{(p\mp\alpha)_\nu}{(p\mp\alpha)^2}\frac{e^{ip_\gamma(x-y)^\gamma}}{(2\pi)^3}. && (3.41)
\end{aligned}$$

3.3.2 Vertices

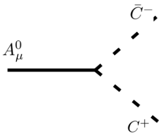
The last thing that we need to calculate in order to consider the perturbative expansion of the partition function is the expressions of the vertices. Given the interaction term (3.23), there are seven different vertices:



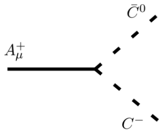
$= i\frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\,;$



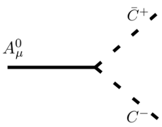
$= \frac{k}{4\pi}p_\mu\,,\ \text{with} \quad p_\mu \neq 0\,;$



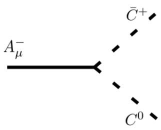
$= \frac{k}{4\pi}(p+\alpha)_\mu\,;$



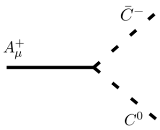
$= -\frac{k}{4\pi}p_\mu\,,\ \text{with} \quad p_\mu \neq 0\,;$



$= -\frac{k}{4\pi}(p-\alpha)_\mu\,;$



$= \frac{k}{4\pi}(p-\alpha)_\mu\,;$



$= -\frac{k}{4\pi}(p+\alpha)_\mu\,.$

Chapter 4

Partition function

In this chapter we want to define and compute, the *partition function*, which is essentially the expectation value of the identity. It has been claimed, [14]-[17], that, up to a normalization, it can be directly related with the topological features of the manifold. In particular, it has been argued that the partition function takes into account the nontrivial states generated by the topology of the manifold.

We use the standard normalization of the partition function, and we define

$$Z = \lim_{N\to\infty} \mathcal{N}^{-1} \int \mathrm{D}\tilde{A}\mathrm{D}\tilde{B}\mathrm{D}\tilde{C}\mathrm{D}\tilde{\bar{C}}\; e^{iS'_{tot}[\tilde{A},\tilde{B},\tilde{C},\tilde{\bar{C}}]}, \tag{4.1}$$

where the normalization $\mathcal{N}$ is

$$\mathcal{N} = \int \mathrm{D}\tilde{A}\mathrm{D}\tilde{B}\mathrm{D}\tilde{C}\mathrm{D}\tilde{\bar{C}}\; e^{iS'_{\Omega\Omega}} e^{iS'_{C\bar{C}}} = (2\pi)^{\frac{N}{2}} \sqrt{\frac{1}{\det\mathbf{\Gamma}}}\,\det\mathbf{\Lambda}\sqrt{\frac{1}{\det\tilde{\mathbf{\Gamma}}}}\,\det\tilde{\mathbf{\Lambda}}.$$

From the equation (3.30) we can deduce:

$$\begin{aligned}
\sqrt{\det\mathbf{\Gamma}} &= (\det\mathbf{\Gamma}^2)^{\frac{1}{4}} = \frac{k}{4\pi}\det\mathbf{\Lambda}; \\
\sqrt{\det\tilde{\mathbf{\Gamma}}} &= (\det\tilde{\mathbf{\Gamma}}^2)^{\frac{1}{4}} = \left(\frac{k}{4\pi}\right)^2 \det\tilde{\mathbf{\Lambda}}.
\end{aligned} \tag{4.2}$$

Therefore, we can write:

$$\mathcal{N} = (2\pi)^{\frac{N}{2}} \left(\frac{4\pi}{k}\right)^3.$$

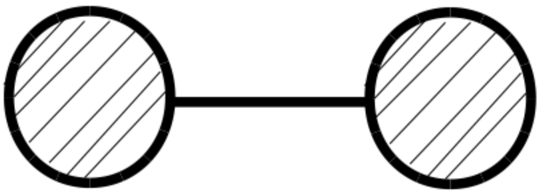
By means of the perturbative expansion of the interaction action, expression (4.1) can be written of the sum of the following terms

$$Z^n = \mathcal{N}^{-1} \int \mathrm{D}\tilde{A}\mathrm{D}\tilde{B}\mathrm{D}\tilde{C}\mathrm{D}\tilde{\bar{C}} \left[\frac{i^n}{n!}(S'_I[A,C,\bar{C}])^n\right] e^{iS'_{\Omega\Omega}[A,B]} e^{iS'_{C\bar{C}}[C,\bar{C}]}. \tag{4.3}$$

The zero order is equal to one, since the normalization that we choose is exactly the functional integral of the quadratic part of the action. The first order is zero because, with the vertices of this theory, there are not vacuum diagrams with one vertex.

Remark

All the diagrams that we can obtain by connecting two generic vacuum diagrams with a propagator



correspond to vanishing contributions. In fact, there are two possibilities which lead to the same result:

- **The propagator in the middle has zero charge**

  Because of the conservation of the momentum the propagator in the figure has to carry $p = 0$, but the zero charge propagator is different from zero if and only if $p \neq 0$;

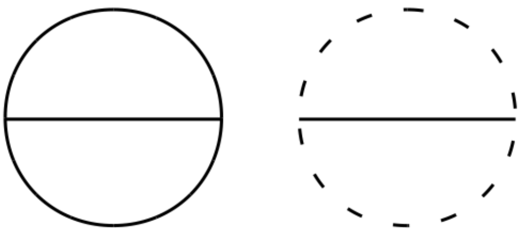
- **The propagator in the middle is charged**

  In this case, the $U(1)$ charge associated with propagator cannot flow through the connecting line of the diagrams because this is not a closed path. Therefore, in this case the diagram has to vanish for the conservation of the charge.

To sum up, we can conclude that only the one-particle-irreducible diagrams give a contribution to the partition function.

4.1 Second order

The one-particle-irreducible diagrams that contribute to this order are all the vacuum two-loop diagrams which we illustrate in the following figure:



(A) (B) (C) (D) (E)

In the above figure the continuous lines represent the bosonic propagators, whereas the dashed lines represent the fermionic propagator. Since in the interaction part of the action there are six different terms relative to the ghost, the antighost and the connection fields, there are six contributions which correspond to the right-sided figure.

With a proper assignment of the momentum to each propagator and each vertex, we obtain

$$(\mathrm{A}) \propto \varepsilon^{\mu\nu\rho}\varepsilon^{\sigma\gamma\tau}\varepsilon^{\mu i\sigma}\varepsilon^{\rho j\tau}\varepsilon^{\nu k\gamma}\sum_{p,q;p\neq q}\frac{(p-q)_j}{(p-q)^2}\frac{(p+\alpha)_i}{(p+\alpha)^2}\frac{(q+\alpha)_k}{(q+\alpha)^2}.$$

The contraction between the five Levi-Civita tensors gives:

$$\begin{aligned}
\varepsilon^{\mu\nu\rho}\varepsilon^{\sigma\gamma\tau}\varepsilon^{\mu i\sigma}\varepsilon^{\rho j\tau}\varepsilon^{\nu k\gamma} &= (\delta^\nu_i\delta^\rho_\sigma - \delta^\nu_\sigma\delta^\rho_i)\varepsilon^{\sigma\gamma\tau}\varepsilon^{\rho j\tau}\varepsilon^{\nu k\gamma} = \\
&= \varepsilon^{\sigma\gamma\tau}\varepsilon^{\sigma j\tau}\varepsilon^{ik\gamma} - \varepsilon^{\sigma\gamma\tau}\varepsilon^{ij\tau}\varepsilon^{\sigma k\gamma} = \\
&= 2\delta^\gamma_j\varepsilon^{ik\gamma} + 2\delta^\tau_k\varepsilon^{ij\tau} = \\
&= 2\varepsilon^{ikj} + 2\varepsilon^{ijk}.
\end{aligned}$$

By means of this substitution, we obtain

$$(\mathrm{A}) \propto \left[\varepsilon^{ikj} - \varepsilon^{ikj}\right]\sum_{p,q;p\neq q}\left[\frac{p_j\alpha_i q_k}{(p-q)^2} - \frac{q_j p_i \alpha_k}{(p-q)^2}\right] = 0.$$

The contribution of the other diagrams are in turn:

$$\begin{aligned}
(B) &\propto \varepsilon^{\mu\nu\rho}\sum_{p,q;p\neq q}\frac{(p+\alpha)_\mu}{(p+\alpha)^2}\frac{(q+\alpha)_\rho}{(q+\alpha)^2}\frac{(p-q)_\nu}{(p-q)^2} = \\
&= \varepsilon^{\mu\nu\rho}\sum_{p,q;p\neq q}\frac{\alpha_\mu q_\rho p_\nu - p_\mu\alpha_\rho q_\nu}{(p+\alpha)^2(q+\alpha)^2(p-q)^2};
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
(C) &\propto \varepsilon^{\mu\nu\rho}\sum_{p,q;p\neq q}\frac{(p-\alpha)_\mu}{(p-\alpha)^2}\frac{(q-\alpha)_\rho}{(q-\alpha)^2}\frac{(p-q)_\nu}{(p-q)^2} = \\
&= \varepsilon^{\mu\nu\rho}\sum_{p,q;p\neq q}\frac{p_\mu\alpha_\rho q_\nu - \alpha_\mu q_\rho p_\nu}{(p-\alpha)^2(q-\alpha)^2(p-q)^2} = 0;
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
(D) &\propto \varepsilon^{\mu\nu\rho}\sum_{p,q;p\neq q}\frac{(p-q)_\mu}{(p-q)^2}\frac{(q-\alpha)_\rho}{(q-\alpha)^2}\frac{(p+\alpha)_\nu}{(p+\alpha)^2} = \\
&= \varepsilon^{\mu\nu\rho}\sum_{p,q;p\neq q}\frac{p_\mu q_\rho\alpha_\nu + \alpha_\mu q_\rho p_\nu}{(p-q)^2(q-\alpha)^2(p+\alpha)^2} = 0;
\end{aligned} \tag{4.6}$$

$$\begin{aligned}(E) &\propto \varepsilon^{\mu\nu\rho} \sum_{p,q;p\neq q} \frac{(q+\alpha)_\mu}{(q+\alpha)^2} \frac{(q-p)_\rho}{(q-p)^2} \frac{(p+\alpha)_\nu}{(p+\alpha)^2} = \\ &= \varepsilon^{\mu\nu\rho} \sum_{p,q;p\neq q} \frac{\alpha_\mu q_\rho p_\nu - q_\mu p_\rho \alpha_\nu}{(q-\alpha)^2 (p-q)^2 (p+\alpha)^2} = 0. \end{aligned} \tag{4.7}$$

Chapter 5

Linking number

In this chapter we carry out the calculations of the self-linking number of some simple framed, oriented and coloured knots and links.

In our formalism it is convenient to decompose the contributions to the Wilson loop which come from the background connection from the remainder. Thus, we rewrite the expression (2.20) as follows

$$W(C) = \mathrm{tr}\left[\mathrm{P}\, e^{i \oint_C \mathrm{d}x^\mu (\alpha_\mu + A_\mu)}\right], \tag{5.1}$$

where C is a knot and we leave out the representation index.

In our presentation of $\mathbb{T}^3$, this manifold is understood as a cube with opposite faces identified. Therefore, each knot C appears as the union of pieces, $C = \cup_j C_j$, in which each component C_j crosses the faces of the cube at most twice. We define *inner knots* those that belong to the interior of a three-ball inside $\mathbb{T}^3$. Inner knots can be represented by knots with no crossing points at all.

By using the decomposition $C = \bigcup_j C_j$, the expression (5.1) becomes

$$W(C) = \mathrm{tr}\left[\mathrm{P}\, e^{i \sum_j \int_{C_j} \mathrm{d}x^\mu (\alpha_\mu + A_\mu)}\right] = \mathrm{tr}\left[\mathrm{P} \prod_j \mathrm{P}\, e^{i \int_{C_j} \mathrm{d}x^\mu (\alpha_\mu + A_\mu)}\right]. \tag{5.2}$$

The path ordering acts nontrivially in each component C_j and in the product of associated holonomies. Since the interior of the cube is simply connected, the constant flat part of the connection α_μ can be locally written as a gauge transformation of the type:

$$\alpha_\mu = -iU^{-1}(x)\partial_\mu U(x) \text{ with } U(x) = e^{i\alpha_\sigma x^\sigma},$$

$$\Rightarrow \alpha_\mu + A_\mu = U^{-1} U A_\mu U^{-1} U - iU^{-1}\partial_\mu U.$$

If we call $\partial C_j^-, \partial C_j^+$ the two edges of the knot piece C_j, respectively at the beginning and at the end with respect to the knot orientation, we can write

$$
\begin{aligned}
W(C) &= \mathrm{tr}\left[\mathrm{P}\prod_{j=1}^{j=n} U^{-1}(\partial C_j^-)\left(\mathrm{P}\, e^{i\int_{C_j}\mathrm{d}x^\mu\left(UA_\mu U^{-1}\right)}\right)U(\partial C_j^+)\right] = \\
&= \mathrm{tr}\Big[U^{-1}(\partial C_1^-)\left(\mathrm{P}\, e^{i\int_{C_1}\mathrm{d}x^\mu\left(UA_\mu U^{-1}\right)}\right)U(\partial C_1^+)\cdots \\
&\qquad \cdots U^{-1}(\partial C_n^-)\left(\mathrm{P}\, e^{i\int_{C_n}\mathrm{d}x^\mu\left(UA_\mu U^{-1}\right)}\right)U(\partial C_n^+)\Big].
\end{aligned}
$$

By using the periodicity of the trace we obtain:

$$
\begin{aligned}
W(C) &= \mathrm{tr}\left[\mathrm{P}\prod_{j=1}^{j=n}\left(P\, e^{i\int_{C_j}\mathrm{d}x^\mu\left(UA_\mu U^{-1}\right)}\right)U(\partial C_j^+)U^{-1}(\partial C_{j+1}^-)\right] = \\
&= \mathrm{tr}\left[\mathrm{P}\prod_{j=1}^{j=n}\left(P\, e^{i\int_{C_j}\mathrm{d}x^\mu\left(UA_\mu U^{-1}\right)}\right)e^{2i\alpha_{\bar\mu}(\partial C_j^+)^{\bar\mu}}\right].
\end{aligned}
\tag{5.3}
$$

Where $\bar\mu$ stands for the direction orthogonal to the crossed surface in which $\partial(C_{j-1}^+)^{\bar\mu} = -\partial(C_j^-)^{\bar\mu}$. As in the standard case we have to introduce a framing in order to fix the ambiguities of the composite operators. Therefore, when it is possible, we define the framing C^f with an arbitrary winding number n in a tubular neighbourhood of the knot. When n is a well defined quantity, we can decompose straightforwardly $C^f = \bigcup_j C_j^f$, where each C_j^f winds n_j times around C_j with the constraint $n = \sum_j n_j$.

As far as the quadratic part in A of the Wilson line operator is concerned, instead of expression (2.26), we obtain

$$
\begin{aligned}
W_r^{(2)}(C)\Big|_f &= \mathrm{tr}\Bigg\{\sum_j\Bigg[\lim_{C_j^f\to C_j}\frac{1}{2}e^{2i\alpha_{\bar\tau}(\partial C_j^+)^{\bar\tau}}\int_{C_j}\mathrm{d}x^\mu\int_{C_j^f}\mathrm{d}y^\rho\times \\
&\quad\times U(x)A_\mu^{(r)}(x)U^{-1}(x)U(y)A_\rho^{(r)}(y)U^{-1}(y)+ \\
&\quad+\sum_{i\neq j}\int_{C_j}\mathrm{d}x^\mu\int_{C_i}\mathrm{d}y^\rho U(x)A_\mu^{(r)}(x)U^{-1}(x)\times \\
&\quad\times\left(\prod_{k=j}^{i}e^{2i\alpha_{\bar\tau}(\partial C_k^+)^{\bar\tau}}\right)U(y)A_\rho^{(r)}(y)U^{-1}(y)\left(\prod_{k=i+1}^{n+j}e^{2i\alpha_{\bar\tau}(\partial C_k^+)^{\bar\tau}}\right)\Bigg]\Bigg\} = \\
&= \mathrm{tr}\Bigg\{\sum_j\Bigg[\lim_{C_j^f\to C_j}\frac{1}{2}e^{2i\alpha_{\bar\tau}(\partial C_j^+)^{\bar\tau}}\int_{C_j}\mathrm{d}x^\mu\int_{C_j^f}\mathrm{d}y^\rho A_\mu^{(r)}(x)U^{-1}(x-y)A_\rho^{(r)}(y)U(x-y)+ \\
&\quad+\left(\prod_{k=1}^{n}e^{2i\alpha_{\bar\tau}(\partial C_k^+)^{\bar\tau}}\right)\sum_{i\neq j}\int_{C_j}\mathrm{d}x^\mu\int_{C_i}\mathrm{d}y^\rho A_\mu^{(r)}(x)\tilde U_{ji}^{-1}(x-y)A_\rho^{(r)}(y)\tilde U_{ji}(x-y)\Bigg]\Bigg\},
\end{aligned}
\tag{5.4}
$$

where k is an integer in $\mathbb{N} \bmod n$ and we define

$$\tilde{U}_{ji}(x-y) = U(x-y)\left(\prod_{k=j}^{i} e^{-2i\alpha_{\bar{\tau}}(\partial C_k^+)^{\bar{\tau}}}\right).$$

Indeed, the expression (5.4) can be written by separating, for each component C_j, the contribution which corresponds to the framed integration from the contributions which correspond to the integration on the other components C_i. Therefore, as far as the expectation value of the expression (5.4) is concern, according to this separation, we write

$$\left\langle W_r^{(2)}(C)\Big|_f\right\rangle = \sum_j \left[\left\langle W_r^{(2)}(C)\Big|_f\right\rangle_{jj} + \sum_{i\neq j}\left\langle W_r^{(2)}(C)\right\rangle_{ij}\right]. \tag{5.5}$$

It is convenient to consider a proper combinations of the algebra generators T^a according to the transformation (3.17). Therefore, we define

$$T^{\pm} = T^1 \pm iT^2 \Rightarrow A_\mu = T^a A_\mu^a = T^3 A_\mu^0 + \frac{1}{\sqrt{2}}T^+ A_\mu^- + \frac{1}{\sqrt{2}}T^- A_\mu^+. \tag{5.6}$$

The commutation rules between these combinations are:

$$[T^3, T^{\pm}] = \pm T^{\pm}\,, \quad [T^+, T^-] = 2T^3. \tag{5.7}$$

For each irreducible representation, the adjoint action of a generic rotation with respect to T^3 on T^b can be written as a rotation of each component in the algebra:

$$\begin{aligned} e^{-i\theta T^3} T^1 e^{i\theta T^3} &= +\cos(\theta)T^1 + \sin(\theta)T^2; \\ e^{-i\theta T^3} T^2 e^{i\theta T^3} &= -\sin(\theta)T^1 + \cos(\theta)T^2 \\ e^{-i\theta T^3} T^3 e^{i\theta T^3} &= T^3, \end{aligned}$$

from which we deduce

$$\begin{aligned} e^{-i\theta T^3} T^+ e^{i\theta T^3} &= e^{-i\theta} T^+; \\ e^{-i\theta T^3} T^- e^{i\theta T^3} &= e^{i\theta} T^-. \end{aligned} \tag{5.8}$$

By means of the above relations, the quantities defined in the expression (5.5) are respectively:

$$\begin{aligned} \left\langle W_r^{(2)}(C)\Big|_f\right\rangle_{jj} &= \frac{1}{2}\lim_{C_j^f \to C_j}\int_{C_j} \mathrm{d}x^\mu \int_{C_j^f} \mathrm{d}y^\rho \Big[\left\langle A_\mu^0(x)A_\rho^0(y)\right\rangle \mathrm{tr}\left(e^{2i\alpha_{\bar{\tau}}(\partial C_j^+)^{\bar{\tau}}} T^3T^3\right) + \\ &+\frac{1}{2}e^{-i\alpha_\tau^0(x-y)^\tau}\left\langle A_\mu^+(x)A_\rho^-(y)\right\rangle \mathrm{tr}\left(e^{2i\alpha_{\bar{\tau}}(\partial C_j^+)^{\bar{\tau}}} T^-T^+\right) + \\ &+\frac{1}{2}e^{i\alpha_\tau^0(x-y)^\tau}\left\langle A_\mu^-(x)A_\rho^+(y)\right\rangle \mathrm{tr}\left(e^{2i\alpha_{\bar{\tau}}(\partial C_j^+)^{\bar{\tau}}} T^+T^-\right)\Big]; \end{aligned} \tag{5.9}$$

$$
\begin{aligned}
\left\langle W_r^{(2)}(C)\right\rangle_{ij} &= \int_{C_j}\mathrm{d}x^\mu\int_{C_i}\mathrm{d}y^\rho\left[\left\langle A_\mu^0(x)A_\rho^0(y)\right\rangle \mathrm{tr}\left(T^3T^3\prod_{k=1}^{n}e^{2i\alpha_{\bar\tau}(\partial C_k^+)^{\bar\tau}}\right)+\right.\\
&+\frac{1}{2}e^{-i\alpha_\tau^0(x-y)^\tau}\left(\prod_{k=j}^{i}e^{2i\alpha_{\bar\tau}^0(\partial C_k^+)^{\bar\tau}}\right)\left\langle A_\mu^+(x)A_\rho^-(y)\right\rangle \mathrm{tr}\left(T^-T^+\prod_{k=1}^{n}e^{2i\alpha_{\bar\tau}(\partial C_k^+)^{\bar\tau}}\right)+\\
&\left.+\frac{1}{2}e^{i\alpha_\tau^0(x-y)^\tau}\left(\prod_{k=j}^{i}e^{-2i\alpha_{\bar\tau}^0(\partial C_k^+)^{\bar\tau}}\right)\left\langle A_\mu^-(x)A_\rho^+(y)\right\rangle \mathrm{tr}\left(T^+T^-\prod_{k=1}^{n}e^{2i\alpha_{\bar\tau}(\partial C_k^+)^{\bar\tau}}\right)\right].
\end{aligned}
\tag{5.10}
$$

5.1 Inner knots

For this class of knots the expression (5.4) greatly simplifies. In fact, in our representation, inner knots have one component inside the cube, thus only the term (5.9) contributes to the quadratic order of the Wilson loop. Moreover, the prefactors dependent on α vanish because there are not crossing points. It remains

$$
\begin{aligned}
\left\langle W_r^{(2)}(C)\Big|_f\right\rangle &= \lim_{C^f\to C}\mathrm{tr}\left\{\frac{1}{2}\left[\int_C\mathrm{d}x^\mu\int_{C^f}\mathrm{d}y^\rho\left\langle A_\mu^0(x)A_\rho^0(y)\right\rangle T^3T^3+\right.\right.\\
&\left.\left.+\frac{1}{2}e^{-i\alpha_\tau^0(x-y)^\tau}\left\langle A_\mu^+(x)A_\rho^-(y)\right\rangle T^-T^+ + \frac{1}{2}e^{i\alpha_\tau^0(x-y)^\tau}\left\langle A_\mu^-(x)A_\rho^+(y)\right\rangle T^+T^-\right]\right\} =\\
&= \lim_{C^f\to C}\mathrm{tr}\left\{\frac{1}{2}\int_C\mathrm{d}x^\mu\int_{C^f}\mathrm{d}y^\rho(-)i\frac{4\pi}{k}\varepsilon^{\mu\nu\rho}\left[\sum_{p_\tau\neq 0}\frac{p_\nu}{p^2}\frac{e^{ip_\tau(x-y)^\tau}}{(2\pi)^3}T^3T^3+\right.\right.\\
&\left.\left.+\frac{1}{2}\sum_{p_\tau}\left(\frac{(p-\alpha)_\nu}{(p-\alpha)^2}\frac{e^{i(p-\alpha)_\tau(x-y)^\tau}}{(2\pi)^3}T^-T^+ + \frac{(p+\alpha)_\nu}{(p+\alpha)^2}\frac{e^{i(p+\alpha)_\tau(x-y)^\tau}}{(2\pi)^3}T^+T^-\right)\right]\right\}.
\end{aligned}
$$

By using the commutation rules (5.7), for each representation r, the quadratic Casimir operator can be written as

$$
C_2(r)\mathbb{I}_{r\times r} = T^aT^a = T^3T^3+\frac{1}{2}T^+T^-+\frac{1}{2}T^-T^+ = T^3T^3+T^+T^- - T^3 = T^3T^3+T^-T^+ + T^3.
$$

By substituting the expression of T^+T^- and T^-T^+, derived from the above identities, we obtain

$$\begin{aligned}
\left\langle W_r^{(2)}(C)\Big|_f \right\rangle &= -\frac{2\pi}{k}\lim_{C^f\to C}\frac{1}{(2\pi)^3}\int_C \mathrm{d}x^\mu \int_{C^f}\mathrm{d}y^\rho \varepsilon^{\mu\nu\rho}\partial_\nu \left[\sum_{p_\tau\neq 0}\frac{e^{ip_\tau(x-y)^\tau}}{p^2}\mathrm{tr}\left(T^3T^3\right)+\right.\\
&\left.+\sum_{p_\tau}\frac{1}{2}\left(\frac{e^{i(p-\alpha)_\tau(x-y)^\tau}}{(p-\alpha)^2}+\frac{e^{i(p+\alpha)_\tau(x-y)^\tau}}{(p+\alpha)^2}\right)\mathrm{tr}\left(C_2(r)\mathbb{I}_{r\times r}-T^3T^3\right)\right]=\\
&= -\frac{2\pi}{k}\lim_{C^f\to C}\frac{\mathrm{d}(r)C_2(r)}{3(2\pi)^3}\int_C \mathrm{d}x^\mu \int_{C^f}\mathrm{d}y^\rho \varepsilon^{\mu\nu\rho}\partial_\nu \times\\
&\times\left[\sum_{p_\tau\neq 0}\frac{e^{ip_\tau(x-y)^\tau}}{p^2}+\sum_{p_\tau}\left(\frac{e^{i(p-\alpha)_\tau(x-y)^\tau}}{(p-\alpha)^2}+\frac{e^{i(p+\alpha)_\tau(x-y)^\tau}}{(p+\alpha)^2}\right)\right]\\
&= -\frac{2\pi}{k}\lim_{C^f\to C}\frac{\mathrm{d}(r)C_2(r)}{3}\int_C \mathrm{d}x^\mu \int_{C^f}\mathrm{d}y^\rho \varepsilon^{\mu\nu\rho}\partial_\nu\phi(x-y). \qquad (5.11)
\end{aligned}$$

By collecting the same factor, in order to obtain an equation similar to equation (2.26), we can define the self-linking number of the knot C as follows

$$\varphi_f(C) = \lim_{C^f\to C}\frac{i}{3}\int_C \mathrm{d}x^\mu \int_{C^f}\mathrm{d}y^\rho \varepsilon^{\mu\nu\rho}\partial_\nu\phi(x-y).$$

A check of the theory is to compare this quantity with its analogue in the case of $\mathbb{R}^3$. In fact, since we can enclose inner knots in an open three-ball, the results have to correspond both to the Gauss linking number between the knot framing and the knot itself.
For a inner knot C, there exists a Seifert surface Σ such that $\partial\Sigma = C$. Thus, we can write:

$$\oint_C \mathrm{d}x^\mu v^\mu = \int_\Sigma \mathrm{d}\sigma^a \varepsilon^{abc}\partial_b v_c.$$

First, let us reduce the computations to the case in which C and its framing form the Hopf link with winding number $+1$. Assuming that $\partial_\nu\phi(x-y)$ is $\mathcal{C}^1$ everywhere in the interior of the cube except for $x=y$, $\varphi_f(K)$ can be expressed as follows

$$\varphi_f(C) = \lim_{C^f\to C} i\frac{1}{3}\lim_{\varepsilon\longrightarrow 0}\int_{0+\varepsilon}^{1-\varepsilon}\mathrm{d}t\,\dot{y}^\rho\int_\Sigma \mathrm{d}\sigma^a(\partial_a\partial_\rho-\delta_\rho^\alpha\partial^2)\phi(x-y). \qquad (5.12)$$

This expression is analogous to the computation of the linking number (2.38), but in this case we do not know the explicit form of $\phi(x-y)$ in the coordinate representation. Let us compute separately the two terms of the integral (5.12) in order to underline their respective features. We start with

$$\begin{aligned}
-\partial^2\phi(x-y) &= -i\delta_\rho^a\left[\delta^3(x-y)-1+2\delta^3(x-y)\cos(\alpha_\gamma(x-y)^\gamma)\right]=\\
&= -i\delta_\rho^a\left[3\delta^3(x-y)-1\right]. \qquad (5.13)
\end{aligned}$$

By integrating the above expression, the term proportional to $\delta^3(x-y)$ vanishes because $x \neq y$ and it remains

$$i\delta^a_\rho \lim_{\varepsilon \longrightarrow 0} \int_\Sigma \mathrm{d}\sigma^a \int_{0+\varepsilon}^{1-\varepsilon} \mathrm{d}t\, \dot{y}^\rho 1 = i \lim_{\varepsilon \longrightarrow 0} \int_\Sigma \mathrm{d}\sigma^\rho \left[y^\rho(1-\epsilon) - y^\rho(0-\epsilon)\right] = 0.$$

The other term can be computed by considering another Seifert surface Σ' of C, in such a way that $\Sigma \cap \Sigma' = \emptyset$; in this case $S = \Sigma \cup \Sigma'$ is a two-dimensional sphere. For each y, $\phi(x-y)$ can be understood as a potential generated in x by a charge in y. From the Gauss theorem, the flow Φ_S of the vector field $\partial_a \phi(x-y)$ through the total surface S is equal to the integral in the enclosed volume τ of the divergence $\partial^2\phi(x-y)$. The total flow is the sum of the two contributions (with the right sign): the flow Φ_Σ plus the flow $\Phi_{\Sigma'}$. Therefore, by using the expression (5.13), we obtain

$$\begin{aligned}
\varphi_f(C) &= \lim_{C^f \to C} -\frac{i}{3} \lim_{\varepsilon \longrightarrow 0} \int_{0+\varepsilon}^{1-\varepsilon} \mathrm{d}t\, \dot{y}^\rho \partial^y_\rho \int_\Sigma \mathrm{d}\sigma^a \partial^x_a \phi(x-y) = \\
&= \lim_{C^f \to C} -\frac{i}{3} \lim_{\varepsilon \longrightarrow 0} [\Phi_S(1-\epsilon) - \Phi_{\Sigma'}(1-\epsilon) - \Phi_S(0+\epsilon) + \Phi_{\Sigma'}(0+\epsilon)] = \\
&= \lim_{C^f \to C} -\frac{i}{3} \lim_{\varepsilon \longrightarrow 0} [\Phi_S(1-\epsilon) - \Phi_S(0+\epsilon)] = -\frac{i^2}{3}[3 - \tau + \tau] = 1. \qquad (5.14)
\end{aligned}$$

This result can be extended to any inner knot with any framing by improving the expression (5.12) so that it takes into account each intersection between the framing and the Seifert surface of the knot. The essential matter is that this quantity comes up to be the Gauss linking number between the knot and its framing, as in the result computed for the knots in $\mathbb{R}^3$.

5.2 The $\pi_1(\mathbb{T}^3)$ generators

Also for this class of knots expression (5.4) simplifies. The representation of these knots has once again one component but, this time, with two crossing point which actually are the same point in the three-torus. There are three types of this kind of knots, one for each spatial direction. By construction, with a chosen direction, the framing has to run in that direction. Moreover, it must have the same orientation (namely it can not be anti-parallel oriented in respect to the knot). Let us chose a direction $C = (x^1, 0, 0)$, with orientation from $-\pi$ to π, and C_f such that $C_f^- = (-\pi, \chi, 0)$, $C_f^+ = (\pi, \chi, 0)$ and it winds n times around C. Therefore, from expression (5.9), we deduce

$$\begin{aligned}
\left\langle W_r^{(2)}(C)\Big|_f\right\rangle &= \frac{1}{2}\lim_{C_j^f\to C_j}\left[\int_{C_j}\mathrm{d}x^\mu\int_{C_j^f}\mathrm{d}y^\rho\left\langle A_\mu^0(x)A_\rho^0(y)\right\rangle\mathrm{tr}\left(e^{2i\alpha_1\pi}T^3T^3\right)+\right.\\
&\quad+\frac{1}{2}e^{-i\alpha_\tau^0(x-y)^\tau}\left\langle A_\mu^+(x)A_\rho^-(y)\right\rangle\mathrm{tr}\left(e^{2i\alpha_1\pi}T^-T^+\right)+\\
&\quad\left.+\frac{1}{2}e^{i\alpha_\tau^0(x-y)^\tau}\left\langle A_\mu^-(x)A_\rho^+(y)\right\rangle\mathrm{tr}\left(e^{2i\alpha_1\pi}T^+T^-\right)\right]=\\
&= -i\frac{2\pi}{k}\lim_{C^f\to C}\int_C\mathrm{d}x^\mu\int_{C^f}\mathrm{d}y^\rho\varepsilon^{\mu\nu\rho}\times\\
&\quad\times\left[v_\nu^1(x-y)\mathrm{tr}\left(e^{2i\alpha_1^0\pi T^3}T^3T^3\right)+v_\nu^2(x-y)\mathrm{tr}\left(e^{2i\alpha_1^0\pi T^3}T^3\right)+\right.\\
&\quad\left.+v_\nu^3(x-y)\mathrm{tr}\left(e^{2i\alpha_1^0\pi T^3}\left(C_2(r)\mathbb{I}_{r\times r}-T^3T^3\right)\right)\right], \qquad (5.15)
\end{aligned}$$

where we define the vector fields $v_\nu^k(x-y)$ with $k=1,2,3$:

$$v_\nu^1(x-y)=\sum_{p_\tau\neq 0}\frac{p_\nu}{p^2}\frac{e^{ip_\tau(x-y)^\tau}}{(2\pi)^3};$$

$$v_\nu^2(x-y)=\sum_{p_\tau}\frac{1}{2(2\pi)^3}\left(\frac{(p-\alpha)_\nu}{(p-\alpha)^2}e^{i(p-\alpha)_\tau(x-y)^\tau}-\frac{(p+\alpha)_\nu}{(p+\alpha)^2}e^{i(p+\alpha)_\tau(x-y)^\tau}\right);$$

$$v_\nu^3(x-y)=\sum_{p_\tau}\frac{1}{2(2\pi)^3}\left(\frac{(p-\alpha)_\nu}{(p-\alpha)^2}e^{i(p-\alpha)_\tau(x-y)^\tau}+\frac{(p+\alpha)_\nu}{(p+\alpha)^2}e^{i(p+\alpha)_\tau(x-y)^\tau}\right).$$

The generator T^3, in the chosen normalization, has the form

$$T^3=\mathrm{diag}[r,r-1,\cdots,1-r,-r],$$

for the representation $r=\frac{1}{2},1,\frac{3}{2},\cdots$. Therefore, the trace of odd powers of T^3 vanishes. Moreover, we can write

$$\mathrm{tr}\left(e^{2i\alpha_1^0\pi T^3}\right)=2\sum_{j=0(\frac{1}{2})}^{r}\cos(2\alpha_1\pi j)=\mathrm{A},$$

$$\mathrm{tr}\left(e^{2i\alpha_1^0\pi T^3}T^3\right)=2\sum_{j=0(\frac{1}{2})}^{r}\sin(2\alpha_1\pi j)j=\mathrm{B},$$

$$\mathrm{tr}\left(e^{2i\alpha_1^0\pi T^3}T^3T^3\right)=2\sum_{j=0(\frac{1}{2})}^{r}\cos(2\alpha_1\pi j)j^2=\mathrm{C},$$

where A, B and C are numbers which only depend on the representation and the background connection.

Once again, we define the self linking number from the expression (5.15) as

$$\begin{aligned}\varphi_f(C) &= \lim_{C^f\to C}\frac{i}{\dim(r)C_2(r)}\int_C \mathrm{d}x^\mu\int_{C^f}\mathrm{d}y^\rho\varepsilon^{\mu\nu\rho}\times\\ &\quad\times\left[v^1_\nu(x-y)\mathrm{C}+v^2_\nu(x-y)\mathrm{B}+v^3_\nu(x-y)(C_2(r)\mathrm{A}-\mathrm{C})\right].\end{aligned}$$

In order to go further in the computation, let us underline that in the integration the vector fields $v^k_\mu(x-y)$ are evaluated only on the points which belong to the knots. Therefore, $\varphi_f(C)$ does not change if we substitute $v^k_\mu(x-y)$ with

$$w^k_\mu(x-y)=\begin{cases}0\,, \text{ if } \sqrt{x_2^2+x_3^2}>\chi\\ v^k_\mu(x-y)\,, \text{ otherwise,}\end{cases}$$

where χ is defined in the explicit parametrization $C_f=(t,-\chi\cos(nt),-\chi\sin(nt))$ that we choose. Actually, we require that $w^k_\mu(x-y)$ decreases smoothly to zero in a neighbourhood of χ, to avoid unessential divergences. Given the rectangle

$$R=\{C\cup[x_1=\pi,0\le x_2\le\epsilon]\cup[-\pi<x_1<\pi,x_2=\epsilon]\cup[x_1=-\pi,0\le x_2\le\epsilon];x_3=0\}\,,$$

with $\epsilon>\chi$ and the orientation fixed by C, we obtain

$$i\lim_{C^f\to C}\int_C\mathrm{d}x^\mu\int_{C^f}\mathrm{d}y^\rho\varepsilon^{\mu\nu\rho}w^1_\mu(x-y)=i\lim_{C^f\to C}\oint_R\mathrm{d}x^\mu\int_{C^f}\mathrm{d}y^\rho\varepsilon^{\mu\nu\rho}w^1_\mu(x-y)=n,$$

where the final result can be obtained with the same steps performed in the previous section, since a Seifert surface can be defined for R. We recall that n is, by definition, the winding number of C_f around C.

The same definitions can be used to compute the contributions of the other vector fields, but, this time, boundary terms have to be subtracted, since these vector fields are not periodic and the circuitation along R is not equal to the integral on C. In fact, for $k=2,3$, we can write

$$\int_C\mathrm{d}x^\mu\int_{C^f}\mathrm{d}y^\rho\varepsilon^{\mu\nu\rho}w^k_\mu(x-y)=\oint_R\mathrm{d}x^\mu\cdots-\int_0^\chi\mathrm{d}x_2\cdots\Big|_{x_1=\pi}-\int_\chi^0\mathrm{d}x_2\cdots\Big|_{x_1=-\pi}.$$

The circuitation along R gives n for $k=3$, whereas it vanishes for $k=2$. The boundary term vanishes by means of the limit $\chi\to 0$. In fact, it can be written as sum of terms of the form

$$\begin{aligned}&-i\sum_{p_\tau}\frac{1}{(2\pi)^3}\sin\left[(p\pm\alpha)_1\pi\right]\int_0^\chi\mathrm{d}x_2e^{i(p\pm\alpha)_2x^2}\times\\ \times&\int_{-\pi}^{\pi}\mathrm{d}t\left[\frac{-n\chi\cos(nt)(p\pm\alpha)_1-(p\pm\alpha)_3}{(p\pm\alpha)^2}e^{-i[(p\pm\alpha)_1t-\chi\cos(nt)(p\pm\alpha)_2-\chi\sin(nt)(p\pm\alpha)_3]}\right],\end{aligned}$$

and the integral in t can be connected to the expressions of the Bessel functions.
To sum up, we obtain

$$\varphi_f(C)=\frac{1}{\dim(r)C_2(r)}\left[n\mathrm{C}+n(C_2(r)\mathrm{A}-\mathrm{C})\right]=\frac{n\mathrm{A}}{\dim(r)}.\tag{5.16}$$

Conclusions

In this thesis we have presented an attempt to compute observables in the Chern-Simons theory defined on $\mathbb{T}^3$. The problem generated by the presence of the zero modes have been discussed in chapter 3. It has been shown that, in a generic nontrivial background, the number of zero modes is reduced from 9 to 3. The strategy proposed in this thesis is to perform the path-integral first in the degrees of freedom with exclusion of the zero modes, and then to complete the computation by integrating on the amplitudes of the zero modes. We concentrate on the first issue of this program.

We classified the space of flat connections on $\mathbb{T}^3$. In the presence of a nontrivial background, the *global* $SU(2)$ symmetry is broken to $U(1)$ and the fields of the theory can accordingly be decomposed. We compute the Feynman propagators by means of the BRST method and rules for the perturbation expansion have been displayed.

The two-loop contribution to the partition function, normalized in a standard way, turn out to be vanishing. It has been demonstrated that all the vacuum-to-vacuum, which are not one-particle-irreducible, do not contribute.

It has been shown that the Wilson line variables can be written in a canonical form which takes into account the presence of the nontrivial background. For knots which belong to a three-ball in $\mathbb{T}^3$, it has been established that the Gauss linking number is obtained by means of the integral of the connection propagator, as in $\mathbb{R}^3$. For generic knots, we compute the $\left(\frac{1}{k}\right)$ contribution to the Wilson line expectation values which turns out to be related again to the Gauss linking number, but in a rather different way; the presence of the background does affect the expectation values.

The natural development of the work of this thesis involve several topics. The validity of the Sorella-Piguet [6] non-normalization theorem, which holds on $\mathbb{R}^3$, could be explored for the Chern-Simons in $\mathbb{T}^3$.

In addiction to the computations of the higher orders in the perturbative expansion, one should take into account also the diagrams with zero modes external legs. in this way, one could evaluate the effects of the integration over the zero modes. This last step is implemented by means of an ordinary integral with respect to a finite number of variables.

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